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The Statistical Analysis of a Random, Moving Surface

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THE STATISTICAL ANALYSIS OF A RANDOM, MOVING SURFACE

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The following statistical properties are derived for a random, moving, Gaussian surface: (1) the probability distribution of the surface elevation and of the magnitude and orientation of the gradient; (2) the average number of zero-crossings per unit distance along a line in an arbitrary direction; (3) the average length of the contours per unit area, and the distribution of their direction; (4) the average density of maxima and minima per unit area of the surface, and the average density of specular points (i.e. points where the two components of gradient take given values); (5) the probability distribution of the velocities of zero-crossings along a given line; (6) the probability distribution of the velocities of contours and of specular points; (7) the probability distribution of the envelope and phase angle, and hence (8) when the spectrum is narrow, the probability distribution of the heights of maxima and minima and the distribution of the intervals between successive zero-crossings along an arbitrary line. All the results are expressed in terms of the two-dimensional energy spectrum of the surface, and are found to involve the moments of the spectrum up to a finite order only. (1), (3), (4), (5) and (6) are discussed in detail for the special case of a narrow spectrum.

The converse problem is also studied and solved: given certain statistical properties of the surface, to find a convergent sequence of approximations to the energy spectrum.

The problems arise in connexion with the statistical analysis of the sea surface.

(More detailed summaries are given at the beginning of each part of the paper.)

CONTENTS

Introduction

On observing waves in the open ocean, one is struck by their irregularity: no single wave retains its identity for long, the distance between neighbouring crests varies with time and place, and frequently it is difficult to assign to the waves any predominant direction or orientation. Thus although the sea surface may, for some purposes, be treated as a uniform train of waves advancing in one direction only, such a representation is usually far from reality.

The first attempt to treat the sea surface as the sum of more than a finite number of simple sine-waves is due to Barber and his collaborators (1946), who used a harmonic analyzer to resolve a length of record, say of wave height or pressure at a fixed point, into its Fourier components. The physical basis for this procedure is that, if the waves are not too steep, the energy in any particular frequency band may be expected to be propagated independently of the rest of the spectrum, and with a velocity characteristic of its frequency. It was shown by Barber & Ursell (1948) that for ocean swell this is in fact nearly true.

Just as sea waves have no single frequency or wavelength, so they have no single direction. One must therefore consider the Fourier spectrum of the sea surface with regard to both frequency and direction or, what is equivalent, the spectrum with regard to wave-number in two horizontal directions. A two-dimensional Fourier analysis for sea waves was proposed by Longuet-Higgins & Barber (1946), who also suggested apparatus for finding a certain amount of information about the spectrum. Independently, Pierson (1952) has emphasized the importance of the distribution of energy with regard to direction when studying the generation and propagation of waves and swell. Thus waves from a limited storm area will decay more or less rapidly with distance according as the spread in direction of the energy is wide or narrow. Similarly, the angular distribution of the energy in a swell will be more or less concentrated according as the region in which it was generated subtends a wide or narrow angle at the point of observation.*

A very interesting problem now arises: the relation between the energy spectrum of the surface and its observable statistical properties. To take a simple example, suppose that we measure the surface elevation ζ at a fixed point: what is the r.m.s. value of ζ with regard to time; what is the average time interval between the maxima of ζ; what proportion of the maxima have heights between two given values?

Questions of this kind have been studied theoretically by several authors, notably by Rice (1944, 1945) in connexion with the analysis of electrical noise currents. Rice considered the function

$$\zeta(t) = \sum_n c_n \cos(\sigma_n t + \epsilon_n), \tag{1}$$

which is the sum of a large number of sine-waves of different frequency $\sigma_n/2\pi$. The phases ϵ_n are random variables distributed uniformly in the interval $(0, 2\pi)$, and the amplitudes c_n are such that in any small interval of σ of width $d\sigma$,

$$\sum_n \tfrac{1}{2} c_n^2 = E(\sigma)\, d\sigma, \tag{2}$$

say (our notation is slightly different from Rice's). The function $E(\sigma)$ may be called the energy spectrum of ζ. It is the cosine transform of G. I. Taylor's correlation function

$$\psi(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \zeta(t')\, \zeta(t'+t)\, dt' \tag{3}$$

* Some other applications of the two-dimensional spectrum may be mentioned. It has been used to calculate the seismic energy generated by sea waves, where the directional distribution of energy is essentially involved (Longuet-Higgins 1950). Eckart has used a two-dimensional analysis to calculate the scattering of sound from the sea surface (1953 *a*) and the waves caused by a random distribution of pressure pulses (1953 *b*). St Denis & Pierson (1953) have applied it to ship motion; see also Cartwright (1956).

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(see Khintchine 1934). One can show that the method of harmonic analysis used by Barber & Ursell (1948) is essentially equivalent to making an estimate of $\sqrt{E}$, within limits of accuracy imposed by the finite length of the record (see Tukey 1949).

Using the above representation, Rice was able to derive many statistical properties of ζ, in particular the probability distribution of ζ itself (which is Gaussian), the average number of zero-crossings of ζ per unit time, the probability distribution of the maxima, and certain statistical properties of the envelope.

It was found by Rudnick (1950) that records of sea-wave pressure are in fact Gaussian (see also Pierson 1952). Barber (1950) considered the distribution of wave heights, that is, the difference in level between a crest and the preceding trough, and compared some observations with the 'random-walk' (or Rayleigh) distribution,* which is the theoretical distribution for a narrow-band spectrum. This distribution has been discussed in more detail by the present author (1952), who showed that the theoretical ratios of the mean wave height, the mean of the highest one-third of the waves, and the height of the highest of N waves were in close agreement with observation. Further observations are given by Watters (1953).

Some two-dimensional statistical properties of the sea surface have also been measured. By photographing the pattern and intensity of sunlight reflected from the sea surface, Cox & Munk (1954*a*, *b*) have deduced the statistical distribution of the two components of surface slope, in winds of different intensity. They find that the distribution differs only slightly from a normal distribution.† One may expect that for swell, which is usually less steep than waves under the action of the wind, the departures from the normal distribution will be still less.

For more than fifty years attempts have been made to construct contour maps of the sea surface. Some results, together with references to earlier work, are given by Schumacher (1952). At the present time some very extensive maps are being made as proposed by Marks (1954). These maps may well be suitable for statistical analysis.

On the theoretical side, Eckart (1946) has considered the intensity of light reflected from a random surface whose gradient and second derivatives are all distributed normally; and he has also calculated the first and second moments of the total curvature. However, no extensive theoretical study of the two-dimensional statistical properties of a random surface appears to have been made.

The purpose of the present paper is to study theoretically the statistical properties of a random, moving Gaussian surface, in relation to its two-dimensional spectrum.

In view of the observations mentioned above, there is reason to believe that some at least of the results are relevant to waves in the open ocean. The analysis may also apply to other geophysical phenomena, for example, to microseisms or perturbations of the ionosphere. In addition, however, the subject is of interest as a branch of geometry, and we shall develop it here on its own account, leaving the application of the results and comparison with observations to a separate study.

* So called because it was derived by Rayleigh in connexion with the theory of sound. See Rayleigh (1880; 1945, pp. 39–42).

† Schooley (1954) has made similar measurements for the river Anacostia. A different technique was used earlier by Duntley (1950) on Lake Winnipeg.

The paper is in three parts. Part I is mainly introductory; we define some convenient parameters for describing the surface: the *long-crestedness*, the *skewness*, the *carrier wave* and the *envelope*, and we find conditions for the surface to split up in various ways into one or more simpler systems of waves.

The chief results are contained in part II. Expressions are derived for the statistical distributions of the surface elevation and the magnitude and direction of the surface slope (§ 2·1); for the average number of zero-crossings of ζ along a line in an arbitrary direction (§ 2·2); for the average length of a given contour and for the distribution of its direction (§ 2·3); for the density of maxima and minima (humps and hollows) per unit area of the surface, and the density of specular points (points where the two components of surface gradient take given values) (§ 2·4); for the statistical distribution of the velocities of the zero-crossings of ζ along a given line (§ 2·5); for the statistical distributions of the velocities of the contours (§ 2·6) and of specular points (§ 2·7). In order to interpret the more complex results, the case when the energy spectrum is narrow, i.e. when the waves are more or less uniform in wavelength and direction, is studied in detail. In § 2·8 some properties of the wave envelope are considered, and from these are deduced the average number of waves in a group, the statistical distribution of the heights of maxima and the distribution of the spacing between successive zeros, all for a narrow spectrum.

In part III the converse problem is considered: given the statistical properties of the surface, to find its energy spectrum. To do this, use is made of a striking feature of the present distributions, that they depend only on the moments of the energy spectrum up to a finite order. Thus the average number of zero-crossings along a line involves only the moments of order 0 and 2. The average number of maxima and minima along a line involves only the moments of order 2 and 4. Properties depending on the motion of the surface involve the odd as well as the even moments. Hence, by considering the statistical properties of the surface along a line in a number of different directions, the moments of the two-dimensional spectrum up to, theoretically, any order can be deduced. From this it is possible to obtain a convergent sequence of approximations to the spectrum (§ 3·3).

Detailed summaries of the results will be found at the beginning of each part.

Part I. Description of the surface

Section 1·1 introduces the representation of a simple wave pattern by a point in the wave-number diagram, and defines the concepts of *carrier wave* and *envelope*, which are afterwards to be extended to a surface with a continuous spectrum. The fundamental definition of a random surface in terms of its spectrum is given in § 1·2.

In § 1·3 conditions are found for the surface to degenerate in various ways. Thus, a simple condition for the surface to be 'long-crested' (i.e. for the energy to travel always in the same direction) is given by (1·3·3). A condition for the surface to consist of no more than two long-crested systems is given by (1·3·7), and a condition for no more than n such systems is given by (1·3·8). All these conditions are expressed in terms of the moments m_{pq} of the energy spectrum E, which are defined by (1·2·7). A condition for E to degenerate into a 'ring' spectrum, i.e. for the energy to have uniform wavelength though not necessarily constant direction, is given by (1·3·11). For standing waves, both (1·3·3) and (1·3·11) must be satisfied simultaneously. Necessary and sufficient conditions for the spectrum to be *narrow* so that the energy is uniform in both wavelength and direction, are given by (1·3·14). Necessary conditions for the existence of not more than two narrow bands of energy are given by (1·3·17) and (1·3·18).

In §1·4 the curve of intersection of the surface by a perpendicular plane in an arbitrary direction θ is considered. It is shown how the spectrum E_θ of this curve is related to the spectrum E of the surface. The *principal direction* is defined as the direction in which the second-order moment of E_θ, and so the r.m.s. wave-number, is a maximum. The minimum r.m.s. wave-number is in the perpendicular direction, and the ratios of the r.m.s. wave-numbers in these two directions is a convenient measure of the *long-crestedness* of the surface.

In §1·5 the *carrier wave* and the *wave envelope* are defined for a surface having a continuous spectrum. It is seen that in general the principal direction of the envelope is different from that of the wave surface, so that the waves form a 'staggered' or echelon pattern. The angle between the two principal directions is called the angle of *skewness*. It is proved that the envelope of the curve in which a vertical plane intersects the surface is the same as the curve in which the plane intersects the envelope.

In §1·6 some special properties of a narrow spectrum are deduced; in particular, that the long-crestedness equals the reciprocal of the r.m.s. angular deviation of energy from the principal direction.

1·1. *The representation of simple wave patterns*

Imagine a single long-crested wave of length λ travelling in a direction which makes an angle θ with the x axis (see figure 1*a*). The wave-number w along a line perpendicular to the crest is defined as

$$w = 2\pi/\lambda. \tag{1·1·1}$$

The wavelength and direction can be specified very conveniently by drawing a vector $\overrightarrow{OP}$ from a fixed point O in a direction θ, such that the length of $\overrightarrow{OP}$ equals w. Then if we consider a section of the surface along any line making an angle θ' with the x axis, it is clear that the wavelength along this section is increased in the ratio $\sec(\theta-\theta')$, so that the wave-number is multiplied by $\cos(\theta-\theta')$. In other words the corresponding wave-number is simply the projection of $\overrightarrow{OP}$ on a line in that direction. In particular, the wave-numbers parallel to the two fixed directions (x, y) are the co-ordinates of the point P with respect of axes in these directions. The equation of the wave surface is then

$$\zeta = c\cos(ux+vy+\sigma t), \tag{1·1·2}$$

where

$$u, v = w\cos\theta,\ w\sin\theta, \tag{1·1·3}$$

and σ is a function of u and v. It will be assumed that σ depends only upon the wavelength, that is on $\sqrt{(u^2+v^2)} = w$;

$$\sigma = \sigma(u,v) = \sigma(w). \tag{1·1·4}$$

We may take σ to be positive, so that the direction of propagation is opposite to $\overrightarrow{OP}$. It follows from (1·1·4) that

$$\sigma(-u, -v) = \sigma(u,v), \tag{1·1·5}$$

that is, waves of the same length but opposite in direction have equal and opposite velocities.

Consider now a pair of long-crested waves of equal amplitude c (figure 1*b*). If these are represented in the wave-number diagram by the vectors $\overrightarrow{OP_1}$ and $\overrightarrow{OP_2}$, where $P_1 = (u_1, v_1)$ and $P_2 = (u_2, v_2)$, we have for the surface elevation

$$\zeta = c\cos(u_1x+v_1y+\sigma_1t)+c\cos(u_2x+v_2y+\sigma_2t). \tag{1·1·6}$$

This may be written

$$\zeta = 2c\cos(u'x+v'y+\sigma't)\cos(\bar{u}x+\bar{v}y+\bar{\sigma}t), \tag{1·1·7}$$

where

$$\left.\begin{aligned} u' &= \tfrac{1}{2}(u_1-u_2), & v' &= \tfrac{1}{2}(v_1-v_2), & \sigma' &= \tfrac{1}{2}(\sigma_1-\sigma_2),\\ \bar{u} &= \tfrac{1}{2}(u_1+u_2), & \bar{v} &= \tfrac{1}{2}(v_1+v_2), & \bar{\sigma} &= \tfrac{1}{2}(\sigma_1+\sigma_2). \end{aligned}\right\} \tag{1·1·8}$$

If the wave-numbers (u_1, v_1) and (u_2, v_2) of the two original waves are nearly equal, the term

$$2c\cos(u'x+v'y+\sigma't) \tag{1·1·9}$$

in (1·1·7) represents a slowly varying amplitude which we may call the 'envelope' and

$$\cos(\bar{u}x+\bar{v}y+\bar{\sigma}t) \tag{1·1·10}$$

represents a 'carrier' wave of approximately the same wavelength and direction as the original waves. The carrier wave is represented in the wave-number diagram by the vector $\vec{OM}$, where M is the mid-point of P_1P_2. The envelope is represented by $\vec{MP_1}$ or $\vec{MP_2}$.

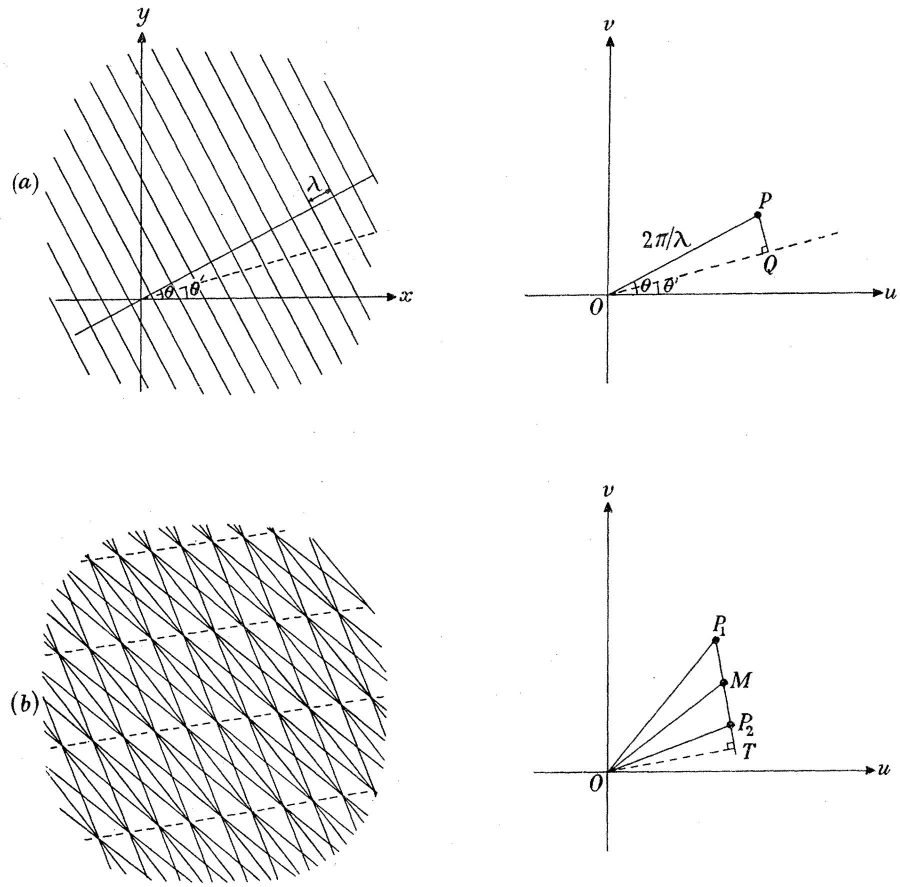

FIGURE 1. Representation in the wave-number plane of (a) a single long-crested wave and (b) the sum of two long-crested waves of different wavelength and direction.

For example, suppose that the two waves are in the same direction but of different wavelength. Then the vectors $\vec{OP_1}$ and $\vec{OP_2}$ are in the same direction and so also are $\vec{OM}$ and $\vec{MP_2}$. Thus the envelope has the same direction as the carrier wave; the crests are infinitely long.

Again, suppose that the two waves are of equal length but in different directions. Then the vectors $\vec{OP_1}$ and $\vec{OP_2}$ are of equal length but different direction. The carrier wave, represented by $\vec{OM}$, lies in the mean direction, but the direction of the envelope is now at right angles to the carrier wave. The result is a short-crested system of waves.

In the general case (figure 1 b) it will be seen that the wave crests are staggered, or form an echelon pattern one behind the other. The direction of this pattern is perpendicular to

P_1P_2. The wave-number perpendicular to P_1P_2 is given by the length OT of the perpendicular from O to P_1P_2; it is the direction in which the wave-numbers of the two component waves are equal.

The angle β between OM and P_1P_2 is a measure of the skewness of the waves.

The envelope (1·1·9) and the carrier (1·1·10) are not necessarily 'free' travelling waves, that is, they do not satisfy equations of the form of (1·1·4). Their representation in the wave-number diagram is valid only so far as the spatial periodicity is concerned. However, for a narrow spectrum $\bar{u}$, $\bar{v}$, $\bar{\sigma}$ are nearly equal to u_1, v_1, σ_1 respectively, and so the carrier wave does move with nearly the free-wave velocity, if the component waves are themselves free waves. But the envelope moves with a velocity whose components are

$$-\left(\frac{\sigma'}{u'}, \frac{\sigma'}{v'}\right) = -\left(\frac{\sigma_1-\sigma_2}{u_1-u_2}, \frac{\sigma_1-\sigma_2}{v_1-v_2}\right). \tag{1·1·11}$$

To a first approximation this is

$$-\left(\frac{\partial\sigma}{\partial u}, \frac{\partial\sigma}{\partial v}\right) = -\left(\frac{\partial w}{\partial u}, \frac{\partial w}{\partial v}\right)\frac{\mathrm{d}\sigma}{\mathrm{d}w} = -(\cos\theta, \sin\theta)\frac{\mathrm{d}\sigma}{\mathrm{d}w}, \tag{1·1·12}$$

which is the so-called group velocity. Thus in this special case the envelope moves with the group velocity of the carrier wave.

1·2. *The representation of a surface having a continuous spectrum*

We now assume that the surface possesses a continuous noise spectrum in two dimensions. Generalizing the representation used in equation (1) we write

$$\zeta(x,y,t) = \sum_n c_n \cos(u_n x + v_n y + \sigma_n t + \epsilon_n), \tag{1·2·1}$$

where it is supposed that the wave-numbers (u_n, v_n) are densely distributed throughout the u, v plane, i.e. there are an infinite number in any elementary area $\mathrm{d}u\,\mathrm{d}v$. σ_n is a function of (u_n, v_n):

$$\sigma_n = \sigma(u_n, v_n); \tag{1·2·2}$$

the amplitudes c_n are random variables such that in any element $\mathrm{d}u\,\mathrm{d}v$ we may assume

$$\sum_n \tfrac{1}{2}c_n^2 = E(u,v)\,\mathrm{d}u\,\mathrm{d}v; \tag{1·2·3}$$

the phases ϵ_n are distributed randomly and with equal probability in the interval $(0, 2\pi)$. The function $E(u,v)$ will be called the energy spectrum of ζ; the mean-square value of ζ per unit area of the sea surface per unit time* is given by

$$\lim_{X,Y,T\to\infty}\frac{1}{8XYT}\int_{-X}^{X}\int_{-Y}^{Y}\int_{-T}^{T}\zeta^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = \sum_n \tfrac{1}{2}c_n^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E(u,v)\,\mathrm{d}u\,\mathrm{d}v. \tag{1·2·4}$$

Thus the contribution to the mean energy from an element $\mathrm{d}u\,\mathrm{d}v$ is proportional to $E\,\mathrm{d}u\,\mathrm{d}v$. We shall write

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}E(u,v)\,\mathrm{d}u\,\mathrm{d}v = m_{00}, \tag{1·2·5}$$

* It is assumed that average values taken with respect to x, y or t are equivalent to average values with respect to the phases ϵ_n.

and in general for the (p, q)th moment of $E(u, v)$ about the origin we write

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u, v)\, u^p v^q \,\mathrm{d}u\,\mathrm{d}v = m_{pq}. \tag{1·2·6}$$

These quantities will occur repeatedly throughout the following analysis. It is assumed that they exist up to all orders required.

The function $E(u, v)$ is closely related to the correlation function $\psi(x, y, t)$ defined by

$$\psi(x, y, t) = \lim_{X, Y, T\to\infty} \frac{1}{8XYT}\int_{-X}^{X}\int_{-Y}^{Y}\int_{-T}^{T} \zeta(x', y', t')\,\zeta(x'+x, y'+y, t'+t)\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}t'. \tag{1·2·7}$$

On substituting from (1·2·1) in the above we find

$$\psi(x, y, t) = \sum_n \tfrac{1}{2}c_n^2 \cos(u_n x + v_n y + \sigma_n t), \tag{1·2·8}$$

which can be written

$$\psi(x, y, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u, v)\cos(ux + vy + \sigma t)\,\mathrm{d}u\,\mathrm{d}v, \tag{1·2·9}$$

so that ψ is the cosine transform of E. The even moments m_{pq} are related very simply to the derivatives of ψ at the origin:

$$m_{pq} = (-1)^r \frac{\partial^{2r}\psi(0, 0, 0)}{\partial x^p\,\partial y^q} \quad (p+q = 2r). \tag{1·2·10}$$

1·3. *Conditions for degeneracy*

Some important features of the surface can be described immediately in terms of the moments. For example, to find a condition that the wave energy shall all travel in one direction, so that the spectrum is effectively one-dimensional, consider the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u_1, v_1)\,E(u_2, v_2)\,(u_1v_2 - u_2v_1)^2\,\mathrm{d}u_1\,\mathrm{d}v_1\,\mathrm{d}u_2\,\mathrm{d}v_2. \tag{1·3·1}$$

If the spectrum is one-dimensional, the product $E(u_1, v_1)\,E(u_2, v_2)$ is zero everywhere except when $u_1/v_1 = u_2/v_2$, when the squared factor vanishes. Therefore the integral vanishes. Conversely, if the spectrum is not one-dimensional there will be a contribution to the integral from some pairs of elements $\mathrm{d}u_1\,\mathrm{d}v_1$, $\mathrm{d}u_2\,\mathrm{d}v_2$ for which $u_1/v_1 \neq u_2/v_2$, and since the integrand is never negative the integral does not vanish. But on expanding the squared factor and separating the integrations with respect to u_1, v_1 and u_2, v_2 we find that (1·3·1) is equal to

$$2(m_{20}m_{02} - m_{11}^2) = 2\begin{vmatrix} m_{20} & m_{11} \\ m_{11} & m_{02}\end{vmatrix} = 2\Delta_2, \tag{1·3·2}$$

say. Thus a necessary and sufficient condition for E to degenerate into a single one-dimensional spectrum is that

$$\Delta_2 = 0. \tag{1·3·3}$$

By similar reasoning, a condition for E to degenerate into two one-dimensional spectra (see figure 2*a*) is that

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} E(u_1, v_1)\,E(u_2, v_2)\,E(u_3, v_3) \times (u_2v_3 - u_3v_2)^2\,(u_3v_1 - u_1v_3)^2\,(u_1v_2 - u_2v_1)^2\,\mathrm{d}u_1\,\mathrm{d}v_1\,\mathrm{d}u_2\,\mathrm{d}v_2\,\mathrm{d}u_3\,\mathrm{d}v_3 \tag{1·3·4}$$

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shall vanish. The squared product may be written

$$\begin{vmatrix} u_1^2 & u_2^2 & u_3^2 \\ u_1 v_1 & u_2 v_2 & u_3 v_3 \\ v_1^2 & v_2^2 & v_3^2 \end{vmatrix}^2 = \epsilon_{ijk} u_i^2 u_j v_j v_k^2 \begin{vmatrix} u_1^2 & u_2^2 & u_3^2 \\ u_1 v_1 & u_2 v_2 & u_3 v_3 \\ v_1^2 & v_2^2 & v_3^2 \end{vmatrix} \qquad (1\cdot3\cdot5)$$

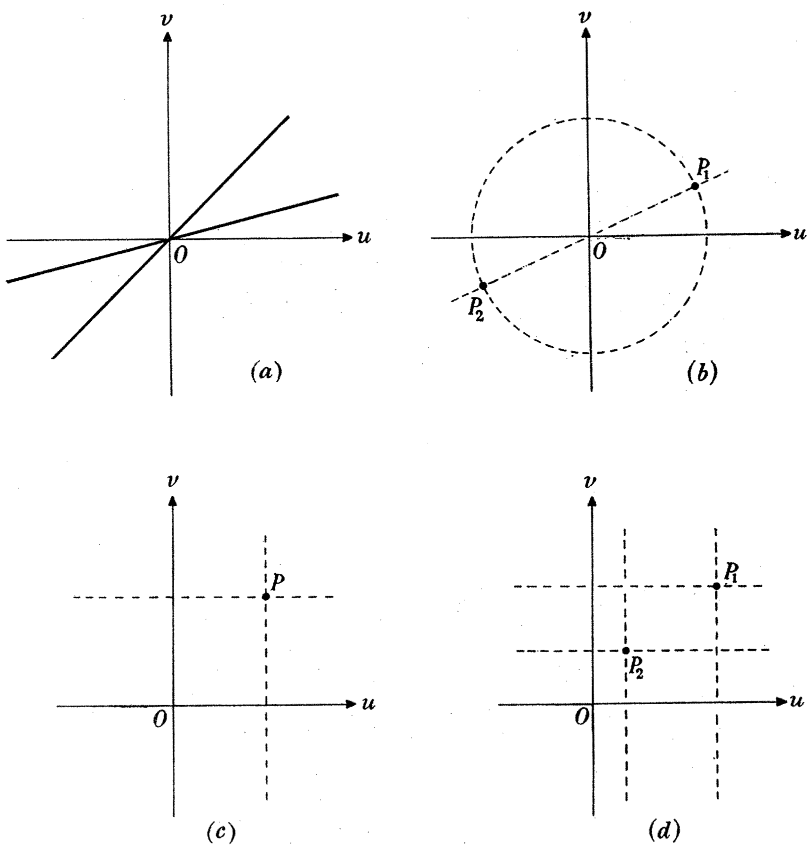

FIGURE 2. The form of the energy spectrum for (*a*) two intersecting long-crested systems of waves, (*b*) a system of standing waves, (*c*) a narrow band of waves, uniform in wavelength and direction and (*d*) two narrow bands of waves.

(where $\epsilon_{ijk} = \pm 1$ according as (i, j, k) is an even or odd permutation of $(1, 2, 3)$, and so the integral equals

$$6 \begin{vmatrix} m_{40} & m_{31} & m_{22} \\ m_{31} & m_{22} & m_{13} \\ m_{22} & m_{13} & m_{04} \end{vmatrix} = 6\Delta_4, \qquad (1\cdot3\cdot6)$$

say. Thus E degenerates into not more than two one-dimensional spectra if and only if

$$\Delta_4 = 0. \qquad (1\cdot3\cdot7)$$

There is an obvious generalization to any number of one-dimensional spectra: the condition that E degenerate into not more than n one-dimensional spectra is that

$$\Delta_{2n} \equiv \begin{vmatrix} m_{2n,0} & m_{2n-1,1} & \cdots & m_{n,n} \\ m_{2n-1,1} & m_{2n-2,2} & \cdots & m_{n-1,n+1} \\ \vdots & \vdots & & \vdots \\ m_{nn} & m_{n-1,n+1} & \cdots & m_{0,2n} \end{vmatrix} = 0. \qquad (1\cdot3\cdot8)$$

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(In practice Δ_2, Δ_4, etc., must be compared with quantities of the same dimensions. Thus Δ_{2n} may be compared with $(m_{20}+m_{02})^n$.)

A condition for E to degenerate into a 'ring' spectrum, such that all the energy corresponds to wave components of the same length but possibly different directions, is that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u_1,v_1)\,E(u_2,v_2)\,[(u_1^2+v_1^2)-(u_2^2+v_2^2)]^2\,du_1\,dv_1\,du_2\,dv_2 \qquad (1\cdot3\cdot9)$$

shall vanish. This integral equals

$$2[(m_{40}+2m_{22}+m_{04})\,m_{00}-(m_{20}+m_{02})^2], \qquad (1\cdot3\cdot10)$$

and so we must have

$$(m_{40}+2m_{22}+m_{04})\,m_{00}-(m_{20}+m_{02})^2=0. \qquad (1\cdot3\cdot11)$$

The condition for the energy to be situated at two diametrically opposite points of the spectrum (giving a standing-wave pattern) is that (1·3·3) and (1·3·11) shall be satisfied simultaneously (see figure 2*b*).

A condition for the energy to be concentrated about a single point in the spectrum is that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u_1,v_1)\,E(u_2,v_2)\,[(u_1-u_2)^2+(v_1-v_2)^2]\,du_1\,dv_1\,du_2\,dv_2 \qquad (1\cdot3\cdot12)$$

shall vanish. This is equivalent to the pair of conditions that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u_1,v_1)\,E(u_2,v_2)\,(u_1-u_2)^2\,du_1\,dv_1\,du_2\,dv_2 \qquad (1\cdot3\cdot13)$$

shall vanish, and a similar integral with factor $(v_1-v_2)^2$. These are the conditions that the energy be concentrated on lines parallel to the v axis and the u axis respectively (see figure 2*c*) On expanding the integrals we have

$$\begin{vmatrix} m_{20} & m_{10} \\ m_{10} & m_{00} \end{vmatrix}=0, \qquad \begin{vmatrix} m_{02} & m_{01} \\ m_{01} & m_{00} \end{vmatrix}=0. \qquad (1\cdot3\cdot14)$$

A condition for the energy to be concentrated about not more than two points in the spectrum (not necessarily opposite) is that

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} E(u_1,v_1)\,E(u_2,v_2)\,E(u_3,v_3)\prod_{i\neq j}[(u_i-u_j)^2+(v_i-v_j)^2]\,du_1\,dv_1\,du_2\,dv_2\,du_3\,dv_3 \quad (1\cdot3\cdot15)$$

shall vanish. The term under the product sign may be written

$$(u_2-u_3)^2\,(u_3-u_1)^2\,(u_1-u_2)^2+(v_2-v_3)^2\,(v_3-v_1)^2\,(v_1-v_2)^2$$
$$+\Sigma(u_2-u_3)^2\,(u_3-u_1)^2\,(v_1-v_2)^2+\Sigma(u_2-u_3)^2\,(v_3-v_1)^2\,(v_1-v_2)^2. \qquad (1\cdot3\cdot16)$$

Since all the terms are non-negative, each separately must vanish. The first two give the conditions

$$\begin{vmatrix} m_{40} & m_{30} & m_{20} \\ m_{30} & m_{20} & m_{10} \\ m_{20} & m_{10} & m_{00} \end{vmatrix}=0, \qquad \begin{vmatrix} m_{04} & m_{03} & m_{02} \\ m_{03} & m_{02} & m_{01} \\ m_{02} & m_{01} & m_{00} \end{vmatrix}=0, \qquad (1\cdot3\cdot17)$$

which are the conditions that the energy shall be at the intersections of two pairs of lines parallel to the v and u axes, i.e. at the corners of a rectangle (figure 2*d*). The remaining

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conditions can also be expressed in terms of the moments. Thus the group of terms under the first summation sign in (1·3·16) leads to the condition

$$4\begin{vmatrix} m_{30} & m_{20} & m_{11} \\ m_{21} & m_{11} & m_{02} \\ m_{20} & m_{10} & m_{01} \end{vmatrix} + 2\begin{vmatrix} m_{30} & m_{20} & m_{22} \\ m_{20} & m_{10} & m_{12} \\ m_{10} & m_{00} & m_{02} \end{vmatrix} + \begin{vmatrix} m_{40} & m_{20} & m_{21} \\ m_{20} & m_{00} & m_{01} \\ m_{21} & m_{01} & m_{02} \end{vmatrix} = 0. \tag{1·3·18}$$

The last group of terms in (1·3·16) leads to a similar condition, the pair of suffixes in each of the moments m_{pq} being interchanged.

We have incidentally shown that each of the combinations of moments on the left-hand sides of equations (1·3·2), (1·3·6), (1·3·11), (1·3·14), (1·3·17) and (1·3·18) is never negative.

1·4. *The spectrum of the surface in an arbitrary direction*

Let us consider the curve in which the surface ζ is intersected by a perpendicular plane in direction θ, that is, the plane $x\sin\theta - y\cos\theta = 0$. The curve will represent a one-dimensional random function, whose spectrum E_θ with regard to the wave-number u' in this direction bears a simple relation to the original spectrum $E(u, v)$. We may call $E_\theta(u')$ the spectrum of the surface in the direction θ.

First, let x', y' denote co-ordinates in the x, y plane in directions parallel and perpendicular to the direction θ:

$$x' = x\cos\theta + y\sin\theta, \quad y' = -x\sin\theta + y\cos\theta. \tag{1·4·1}$$

Reciprocally, x, y are given in terms of x', y' by similar relations, but with the sign of θ reversed. On substituting in (1·2·1) we have

$$\zeta = \sum_n c_n \cos(u_n' x' + v_n' y' + \sigma_n' t + \epsilon_n), \tag{1·4·2}$$

where
$$u_n' = u_n\cos\theta + v_n\sin\theta, \quad v_n' = -u_n\sin\theta + v_n\cos\theta, \tag{1·4·3}$$

that is, the new wave-number u_n' is the co-ordinate, in the direction θ, of the point (u_n, v_n), and v_n' is the co-ordinate at right angles. We have also

$$\sigma_n' = \sigma\{\surd(u_n^2 + v_n^2)\} = \sigma\{\surd(u_n'^2 + v_n'^2)\}. \tag{1·4·4}$$

On the curve of intersection we have $y' = 0$ and so

$$\zeta = \sum_n c_n \cos(u_n' x' + \sigma_n' t + \epsilon_n). \tag{1·4·5}$$

The spectrum $E_\theta(u')$ of this curve is defined as the function such that the energy corresponding to any small interval $(u', u' + \mathrm{d}u')$ is $E_\theta(u')\,\mathrm{d}u'$. Thus if $\sum\limits_{\mathrm{d}u', v'}$ denotes summation over the strip $(u', u' + \mathrm{d}u')$,

$$E_\theta(u')\,\mathrm{d}u = \sum_{\mathrm{d}u', v'} \tfrac{1}{2}c_n^2 = \mathrm{d}u' \int_{-\infty}^{\infty} E(u, v)\,\mathrm{d}v', \tag{1·4·6}$$

and therefore
$$E_\theta(u') = \int_{-\infty}^{\infty} E(u, v)\,\mathrm{d}v'. \tag{1·4·7}$$

In other words, if we take a section of the surface in any direction θ, the spectrum $E_\theta(u')$ of this section is found by integrating $E(u, v)$ along the line through $P = (u'\cos\theta, u'\sin\theta)$ at right angles to $\overrightarrow{OP}$.

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From equation (1·4·7) there follow some simple and fundamental relations between the moments of the spectrum $E_\theta(u')$ and the moments of the original distribution $E(u,v)$. Let the nth moment of E_θ about the origin be denoted by $m_n(\theta)$. Then we have

$$m_n(\theta) = \int_{-\infty}^{\infty} E_\theta(u')\, u'^n \mathrm{d}u' = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u,v)\, u'^n \mathrm{d}u'\,\mathrm{d}v'. \tag{1·4·8}$$

Since
$$u' = u\cos\theta + v\sin\theta, \quad v' = -u\sin\theta + v\cos\theta \tag{1·4·9}$$

and
$$\frac{\partial(u',v')}{\partial(u,v)} = 1, \tag{1·4·10}$$

we have
$$m_n(\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u,v)\,(u\cos\theta+v\sin\theta)^n\,\mathrm{d}u\,\mathrm{d}v. \tag{1·4·11}$$

After expanding the binomial and integrating each term we find

$$m_n(\theta) = m_{n,0}\cos^n\theta + \binom{n}{1} m_{n-1,1}\cos^{n-1}\theta\sin\theta + \ldots + m_{0,n}\sin^n\theta, \tag{1·4·12}$$

where m_{pq} is the (p,q)th moment of E about the origin (equation (1·2·6)) and $\binom{n}{r}$ denotes the binomial coefficient.

In particular we have
$$m_0(\theta) = m_{00}, \tag{1·4·13}$$

showing that the r.m.s. value of $\zeta(x')$ is independent of the direction θ and equals the r.m.s. value of $\zeta(x,y)$. Next,
$$m_1(\theta) = m_{10}\cos\theta + m_{01}\sin\theta. \tag{1·4·14}$$

If $(\bar{u},\bar{v})$ denotes the centroid of the two-dimensional spectrum:

$$(\bar{u},\bar{v}) = \left(\frac{m_{10}}{m_{00}}, \frac{m_{01}}{m_{00}}\right), \tag{1·4·15}$$

and if $\bar{u}'$ denotes the mean wave-number of the spectrum of $\zeta(x')$ we have

$$\bar{u}' = \frac{m_1(\theta)}{m_0(\theta)} = \bar{u}\cos\theta + \bar{v}\sin\theta, \tag{1·4·16}$$

which can be expressed as
$$\bar{u}' = \bar{w}\cos(\theta - \bar{\theta}), \tag{1·4·17}$$

where
$$(\bar{u},\bar{v}) = (\bar{w}\cos\bar{\theta}, \bar{w}\sin\bar{\theta}). \tag{1·4·18}$$

$\bar{w}$ and $\bar{\theta}$ may be called the mean wave-number and mean direction of the two-dimensional spectrum. Thus the mean wave-number of $E_\theta(u')$ is the projection of the mean wave-number of $E(u,v)$ on to the line of the section. The physical significance of this result will become clearer in § 1·5.

The second moment $m_2(\theta)$ is particularly important. From (1·4·12) we have

$$m_2(\theta) = m_{20}\cos^2\theta + 2m_{11}\cos\theta\sin\theta + m_{02}\sin^2\theta. \tag{1·4·19}$$

The maxima and minima of this expression are given by

$$m_{2\,\mathrm{max.}}, m_{2\,\mathrm{min.}} = \tfrac{1}{2}[(m_{20}+m_{02}) \pm \surd\{(m_{20}-m_{02})^2 + 4m_{11}^2\}], \tag{1·4·20}$$

and these occur always in two directions at right angles, given by

$$\tan 2\theta_p = \frac{2m_{11}}{m_{20}-m_{02}}. \tag{1·4·21}$$

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If θ_p corresponds to the maximum we have

$$m_2(\theta) = m_{2\,\max.}\cos^2(\theta-\theta_p)+m_{2\,\min.}\sin^2(\theta-\theta_p). \tag{1·4·22}$$

The direction θ_p corresponding to the maximum will be called the *principal direction* of the waves. Now

$$\left(\frac{m_2(\theta)}{m_{00}}\right)^{\frac{1}{2}} \tag{1·4·23}$$

is the r.m.s. wave-number in the direction θ. For a long-crested system of waves the r.m.s. wave-number is a maximum perpendicular to the crests and a minimum parallel to the crests. In general, therefore, a convenient measure of the *long-crestedness* is given by the ratio

$$\left(\frac{m_{2\,\max.}}{m_{2\,\min.}}\right)^{\frac{1}{2}}, \tag{1·4·24}$$

which we denote by $1/\gamma$. Thus we have

$$\gamma^2 = \frac{m_{2\,\min.}}{m_{2\,\max.}} = \frac{(m_{20}+m_{02})-\sqrt{\{(m_{20}-m_{02})^2+4m_{11}^2\}}}{(m_{20}+m_{02})+\sqrt{\{(m_{20}-m_{02})^2+4m_{11}^2\}}}. \tag{1·4·25}$$

When the condition (1·3·3) for a one-dimensional spectrum is satisfied we have

$$\gamma = 0, \quad 1/\gamma = \infty. \tag{1·4·26}$$

The two quantities $m_{2\,\max.}$, $m_{2\,\min.}$ are clearly invariant under a rotation of the axes. Hence we have also the invariants

$$m_{2\,\max.}+m_{2\,\min.} = m_{20}+m_{02} = m, \tag{1·4·27}$$

say, and

$$m_{2\,\max.}\,m_{2\,\min.} = m_{20}m_{02}-m_{11}^2 = \Delta_2. \tag{1·4·28}$$

1·5. *The wave envelope*

By analogy with §1·1 we define the mean wave-number as the centroid of the energy distribution:

$$\left.\begin{aligned} m_{00}\,\bar{u} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u,v)\,u\,\mathrm{d}u\,\mathrm{d}v = m_{10},\\ m_{00}\,\bar{v} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u,v)\,v\,\mathrm{d}u\,\mathrm{d}v = m_{01}\end{aligned}\right\} \tag{1·5·1}$$

and we define also the mean frequency $\bar{\sigma}/2\pi$ by the analogous equation

$$m_{00}\,\bar{\sigma} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u,v)\,\sigma\,\mathrm{d}u\,\mathrm{d}v = m'_{00}, \tag{1·5·2}$$

say. Now let (1·2·1) be written in the form

$$\zeta = \mathscr{R}\sum_n c_n \exp\{\mathrm{i}(u_n x+v_n y+\sigma_n t+\epsilon_n)\} \tag{1·5·3}$$

$$= \mathscr{R}[\sum_n c_n \exp\{\mathrm{i}[(u_n-\bar{u})\,x+(v_n-\bar{v})\,y+(\sigma_n-\bar{\sigma})\,t+\epsilon_n]\}\exp\{\mathrm{i}(\bar{u}x+\bar{v}y+\bar{\sigma}t)\}], \tag{1·5·4}$$

where $\mathscr{R}$ denotes the real part. This expresses ζ as the product of a carrier wave

$$\exp\{\mathrm{i}(\bar{u}x+\bar{v}y+\bar{\sigma}t)\}, \tag{1·5·5}$$

and a slowly varying amplitude function

$$\rho\,\mathrm{e}^{\mathrm{i}\phi} = \sum_n c_n \exp\{\mathrm{i}[(u_n-\bar{u})\,x+(v_n-\bar{v})\,y+(\sigma_n-\bar{\sigma})\,t+\epsilon_n]\}, \tag{1·5·6}$$

which may be called the complex envelope. (ρ and ϕ are real functions of (x, y, t), with $\rho \geqslant 0$.) Any other choice for the frequency of the carrier wave might have been taken; the mean wave-number has the unique property that the secular increase of ϕ with x and y is zero (as will be shown in § 2·8).

Comparing (1·5·3) and (1·5·4) we see that *the real part* of the amplitude function, i.e. $\rho\cos\phi$, has the same spectrum as ζ, only with the origin moved to $(\bar{u}, \bar{v})$; similarly for the imaginary part. (ρ itself, however, is a different type of function, being essentially positive.) The properties of the envelope, therefore, are defined by the moments of the energy distribution about the mean. Let

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u,v)\,(u-\bar{u})^p\,(v-\bar{v})^q\,\mathrm{d}u\,\mathrm{d}v = \mu_{pq}. \tag{1·5·7}$$

It is easily seen that

$$\mu_{00} = m_{00}, \quad \mu_{10} = \mu_{01} = 0, \tag{1·5·8}$$

and

$$\left.\begin{aligned} \mu_{20} &= m_{20} - \bar{u}^2 m_{00} = (m_{20}m_{00} - m_{10}^2)/m_{00}, \\ \mu_{11} &= m_{11} - \bar{u}\bar{v}m_{00} = (m_{11}m_{00} - m_{10}m_{01})/m_{00}, \\ \mu_{02} &= m_{02} - \bar{v}^2 m_{00} = (m_{02}m_{00} - m_{01}^2)/m_{00}. \end{aligned}\right\} \tag{1·5·9}$$

The second moment about the mean in a direction θ is

$$\mu_2(\theta) = \mu_{20}\cos^2\theta + 2\mu_{11}\cos\theta\sin\theta + \mu_{02}\sin^2\theta, \tag{1·5·10}$$

and the principal direction of the envelope is given by

$$\tan 2\theta_e = \frac{2\mu_{11}}{\mu_{20} - \mu_{02}}. \tag{1·5·11}$$

The angle β between the principal direction of the envelope and the principal direction of the waves is given by

$$\tan 2\beta = \tan 2(\theta_e - \theta_p) = \frac{2\mu_{11}(m_{20} - m_{02}) - 2m_{11}(\mu_{20} - \mu_{02})}{(\mu_{20} - \mu_{02})\,(m_{20} - m_{02}) + 4\mu_{11}m_{11}}. \tag{1·5·12}$$

Thus β is a convenient measure of the *skewness* of the waves (see § 1·1).

Consider again the curve of intersection of the surface with a vertical plane in direction θ. We may see that the envelope of this curve is simply the intersection of the two-dimensional envelope with the vertical plane. For on the one hand we have from (1·5·6)

$$\rho\,\mathrm{e}^{\mathrm{i}\phi} = \sum_n c_n \exp\{\mathrm{i}[(u_n' - \bar{u}')\,x' + (v_n' - \bar{v})\,y' + (\sigma_n' - \bar{\sigma})\,t + \epsilon_n]\}, \tag{1·5·13}$$

where u_n' and v_n' are given by (1·4·3) and

$$\bar{u}' = \bar{u}\cos\theta + \bar{v}\sin\theta, \quad \bar{v}' = -\bar{u}\sin\theta + \bar{v}\cos\theta. \tag{1·5·14}$$

The intersection of the envelope by the plane $y' = 0$ is therefore given by

$$\rho\,\mathrm{e}^{\mathrm{i}\phi} = \sum_n c_n \exp\{\mathrm{i}[(u_n' - \bar{u}')\,x' + (\sigma_n' - \bar{\sigma})\,t + \epsilon_n]\}. \tag{1·5·15}$$

On the other hand from (1·4·5) we may write

$$\zeta(x', t) = \mathscr{R}[\sum_n c_n \exp\{\mathrm{i}[(u_n' - \bar{u})\,x' + (\sigma_n' - \bar{\sigma})\,t + \epsilon_n]\}\exp\{\mathrm{i}(\bar{u}'x' + \bar{\sigma}t)\}], \tag{1·5·16}$$

where $\bar{u}'$ is given by (1·5·14). But we saw in § 1·4 that $\bar{u}'$ is also the mean wave-number for the function $\zeta(x', t)$ and therefore $\exp\{i(\bar{u}'x' + \bar{\sigma}t)\}$ is, by definition, the carrier wave for $\zeta(x', t)$ and (1·5·15) is the envelope; which proves the result.

1·6. *A narrow spectrum*

A case of special interest is when the energy is concentrated near a single point in the spectrum, so that the component waves are nearly constant in wavelength and direction. As we saw earlier, the conditions satisfied by the first-order and second-order moments are that the left-hand sides of equations (1·3·14) are small. In terms of the moments this implies $\mu_{20} + \mu_{02} \ll m_{20} + m_{02}$, or equivalently

$$\mu_{20} + \mu_{02} \ll (\bar{u}^2 + \bar{v}^2)\, m_{00}. \tag{1·6·1}$$

The envelope of the waves, as defined in the previous section, then has some special properties. If in (1·5·2) we expand $\sigma(u, v)$ in a Taylor series about $(\bar{u}, \bar{v})$ we have

$$\begin{aligned} m_{00}\bar{\sigma} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u, v)\left[\sigma(\bar{u}, \bar{v}) + (u - \bar{u})\frac{\partial}{\partial \bar{u}}\sigma(\bar{u}, \bar{v}) + (v - \bar{v})\frac{\partial}{\partial \bar{v}}\sigma(\bar{u}, \bar{v})\right] \mathrm{d}u\,\mathrm{d}v \\ &= m_{00}\sigma(\bar{u}, \bar{v}) + \mu_{10}\frac{\partial}{\partial \bar{u}}\sigma(\bar{u}, \bar{v}) + \mu_{01}\frac{\partial}{\partial \bar{v}}\sigma(\bar{u}, \bar{v}), \end{aligned} \tag{1·6·2}$$

terms of higher order being negligible. Since $\mu_{10} = \mu_{01} = 0$ we have

$$\bar{\sigma} = \sigma(\bar{u}, \bar{v}). \tag{1·6·3}$$

In other words, the carrier wave is a free wave with the frequency and velocity appropriate to its wave-number. Further, in (1·5·6) we may write

$$\sigma_n - \bar{\sigma} = (u_n - \bar{u})\frac{\partial \bar{\sigma}}{\partial \bar{u}} + (v_n - \bar{v})\frac{\partial \bar{\sigma}}{\partial \bar{v}}, \tag{1·6·4}$$

so that

$$\rho\, e^{i\phi} = \sum_n \exp\{i[(u_n - \bar{u})(x + t\partial\bar{\sigma}/\partial\bar{u}) + (v_n - \bar{v})(y + t\partial\bar{\sigma}/\partial\bar{v})]\}, \tag{1·6·5}$$

which is a function of $(x + t\partial\bar{\sigma}/\partial\bar{u})$ and $(y + t\partial\bar{\sigma}/\partial\bar{v})$ only. In other words, the envelope moves bodily with velocity

$$\left(-\frac{\partial \bar{\sigma}}{\partial \bar{u}}, -\frac{\partial \bar{\sigma}}{\partial \bar{v}}\right). \tag{1·6·6}$$

σ being a function of $w = (u^2 + v^2)^{\frac{1}{2}}$ only, this velocity is

$$-\left(\frac{\partial \bar{w}}{\partial \bar{u}}, \frac{\partial \bar{w}}{\partial \bar{v}}\right)\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\bar{w}} = -(\cos\bar{\theta}, \sin\bar{\theta})\frac{\mathrm{d}\bar{\sigma}}{\mathrm{d}\bar{w}}, \tag{1·6·7}$$

which is the group velocity of the carrier wave.

Let axes be chosen so that the u axis passes through the centroid $(\bar{u}, \bar{v})$, making $\bar{v} = 0$. On expanding u^p, $= \{\bar{u} + (u - \bar{u})^p\}$, by the binomial theorem we have

$$\begin{aligned} m_{pq} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u, v)\, u^p v^q\, \mathrm{d}u\,\mathrm{d}v \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u, v)\, [\bar{u}^p + p\bar{u}^{p-1}(u - \bar{u}) + \ldots + (u - \bar{u})^p]\,(v - \bar{v})^q\, \mathrm{d}u\,\mathrm{d}v \\ &= \bar{u}^p\mu_{00} + p\bar{u}^{p-1}\mu_{1q} + \ldots + \mu_{pq}. \end{aligned} \tag{1·6·8}$$

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In particular, since $\mu_{10} = \mu_{01} = 0$, we have

$$m_{20} = \bar{u}^2\mu_{00} + \mu_{20}, \quad m_{11} = \mu_{11}, \quad m_{02} = \mu_{02}. \tag{1·6·9}$$

Thus (1·4·9) becomes

$$m_2(\theta) = \bar{u}^2\mu_{00}\cos^2\theta + (\mu_{20}\cos^2\theta + 2\mu_{11}\cos\theta\sin\theta + \mu_{02}\sin^2\theta). \tag{1·6·10}$$

Since $\mu_{11} \leqslant (\mu_{20}\mu_{02})^{\frac{1}{2}}$ it follows that all three moments $\mu_{20}, \mu_{11}, \mu_{02}$ are small compared with $\bar{u}^2\mu_{00}$. Hence $m_2(\theta)$ has a maximum near $\theta = 0, \pi$ and a minimum near $\theta = \pm\frac{1}{2}\pi$. In other words, the principal direction lies along the axis of u. The long-crestedness γ^{-1} was defined as the ratio of the r.m.s. wave-numbers parallel and perpendicular to the principal directions. Thus

$$\gamma^2 = \frac{m_2(\frac{1}{2}\pi)}{m_2(0)} = \frac{\mu_{02}}{\bar{u}^2\mu_{00}}. \tag{1·6·11}$$

Now in the neighbourhood of the centroid we have $v = \bar{u}\theta$ very nearly, so that

$$\mu_{02} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u,v)\,\bar{u}^2\theta^2\,\mathrm{d}u\,\mathrm{d}v. \tag{1·6·12}$$

Hence

$$\gamma^2\mu_{00} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u,v)\,\theta^2\,\mathrm{d}u\,\mathrm{d}v. \tag{1·6·13}$$

In other words, *γ is the r.m.s. angular deviation of the energy from the mean direction.*

Since the principal direction of the waves coincides with the u axis, the angle of skewness β is the angle between the u axis and the principal direction of the envelope, that is,

$$\tan 2\beta = \frac{2\mu_{11}}{\mu_{20} - \mu_{02}}. \tag{1·6·14}$$

It will be found convenient to introduce one further parameter for a narrow wave spectrum:

$$\nu = (\mu_{20}/\bar{u}^2\mu_{00})^{\frac{1}{2}}. \tag{1·6·15}$$

ν is proportional to the r.m.s. width of the spectrum in the principal direction. We shall show in § 2·8 that ν^{-1} is a measure of the average number of waves per 'group'.

Part II. Statistical properties

The fundamental statistical distributions of ζ and its derivatives are given in §2·1. The following three sections are devoted to properties of the surface not involving motion, and the next three sections to the distributions of velocities associated with these properties. Lastly, §§2·8 to 2·10 deal with the envelope of the surface and with properties which can be derived from it.

The distributions of the surface elevation ζ and of the two components of gradient $\partial\zeta/\partial x$, $\partial\zeta/\partial y$ are normal in one and two dimensions respectively (equations (2·1·8) and (2·1·12)). The greatest r.m.s. gradient is in the principal direction of the surface. The distribution of the magnitude α of the gradient regardless of direction is given by (2·1·31) and figure 3. For very short-crested waves the distribution is a Rayleigh distribution; for very long-crested waves it tends to a normal distribution, with an anomaly near $\alpha = 0$, the shape of which is shown in figure 4. The probability distribution of the horizontal direction θ of the gradient is given by (2·1·37) and figure 5. It is shown that as the long-crestedness increases, the direction of the gradient becomes more and more certain to be near the principal direction.

In §2·2 is found the mean number of zeros of the surface along a horizontal line in an arbitrary direction θ. The number N_0 per unit distance is given by (2·2·5). Thus N_0 is a maximum when θ is in the principal direction, and a minimum in the direction at right angles. The ratio $N_{0\,\mathrm{max.}}/N_{0\,\mathrm{min.}}$

is equal to the long-crestedness γ^{-1}. The mean number of times that the surface crosses a line at arbitrary level is also found (2·2·12), and the mean number of crests and troughs of a plane section of the surface in any direction.

The average length of a contour of constant height drawn on the surface is derived in §2·3. The length $\bar{s}$ per unit area is given by (2·3·16). The distribution of the direction θ of the normal to a particular contour, at points distributed uniformly along it, is given by (2·3·23). As in §2·1, when the waves become long-crested, the direction becomes concentrated near the principal direction.

Next (§2·4) the average density of maxima and minima of the surface per unit horizontal area is considered. It is shown that the average density of maxima, $D_{\text{ma.}}$, is equal to one-half of the average density of saddle-points, and to one-quarter of the total density of stationary points on the surface. The actual density is given by (2·4·51), in the general case. For a narrow spectrum, the density is given by (2·4·61) and table 1. $D_{\text{ma.}}$ depends not only on the long-crestedness but also on a parameter a representing the *peakedness* of the energy distribution with regard to direction.

Passing now to properties depending on the motion of the surface, we consider in §2·5 the velocity of the zeros of the surface along an arbitrary line. We find that the velocities have a probability distribution given by (2·5·15). This is symmetrical about a mean value depending on the first-order moments. Similarly, the velocities of maxima and minima of a plane section of the surface have a distribution given by (2·5·19). These distributions are studied in the special case of a narrow spectrum. The width of the distribution depends on both the width of the energy spectrum and on the dispersive properties of the medium.

The motion of a contour on the surface can be defined locally by the velocities of its points of intersection with lines parallel to the axes of x, y (§2·6). The distribution of the reciprocals of the velocities, which is simpler than that of the velocities themselves, is given by (2·6·21). The distribution is discussed in detail for the case of a narrow spectrum; the contours of constant probability are then concentric ellipses.

In §2·7 is considered the motion of the 'specular points' of the surface, that is, points where the gradient of the surface takes a certain value. (Such points on the sea surface are, to a distant observer, points of reflected sunlight.) The probability distribution of the two components of velocity is given by (2·7·31). In the special case of a narrow spectrum the mean velocity of the specular points is equal to the phase velocity of the carrier wave. The departures of the velocities from the mean velocity have a distribution given by (2·7·37). This expression has been computed for three different values of the peakedness a, and is shown in figure 12 a, b and c.

In §2·8 we consider some properties of the wave envelope, from which we derive some other useful distributions. The distribution of the envelope function itself is a Rayleigh distribution (2·8·6). The joint distribution of ρ, $\partial\rho/\partial x$ and $\partial\rho/\partial y$ is given by (2·8·15), from which it follows that the envelope possesses a number of properties analogous to the original surface. The envelope also controls the 'grouping' of the waves, and we find, taking a section of the surface in an arbitrary direction θ, that the average length of a group is $2/\mathbf{N}$, where $\mathbf{N}$ is given by (2·8·26). Hence the average length of a group is least in the principal direction and greatest in the direction at right angles. We find the average number of waves in a group (2·8·27) and the condition that this shall be independent of the direction θ (2·8·28).

When the spectrum is narrow, the crests of the waves lie practically on the envelope, and so we are able to deduce that the probability distribution of the heights of crests is approximately a Rayleigh distribution (2·9·1). The distribution of the heights of maxima is found through the distribution of the heights of the maxima of the envelope (2·9·8). This distribution is shown in figure 13 for different values of peakedness a. The limiting case of two crossing swells ($a=1$) is given by (2·8·12) and is also shown in figure 13.

Finally, in §2·10 is deduced the distribution of the intervals l between successive zero-crossings, or between the successive points of intersection of a straight line with a contour at fixed height. The distribution of l for waves of all heights is given by (2·10·18). However, if the waves are classified according to their height, the distribution of l is given by (2·10·23), and hence it is found that l is less scattered for the high waves than for the low waves. The degree of scattering is inversely proportional to the average number of waves in a group.

2·1. *The distribution of surface elevation and gradient*

Let $\xi_1, \ldots, \xi_n$ be n quantities, each the sum of a large number of independent variables whose expectation is zero. Then under certain general conditions (discussed by Rice 1944, 1945; see also Cramér 1937) the joint-probability distribution of $\xi_1, \ldots, \xi_n$ is normal in n dimensions:

$$p(\xi_1, \ldots, \xi_n) = \frac{1}{(2\pi)^{\frac{1}{2}n}\Delta^{\frac{1}{2}}} \exp\{-\tfrac{1}{2}M_{ij}\xi_i\xi_j\}, \tag{2·1·1}$$

where (M_{ij}) is the inverse matrix to

$$(\Xi_{ij}) = \begin{pmatrix} \overline{\xi_1^2} & \overline{\xi_1\xi_2} & \cdots & \overline{\xi_1\xi_n} \\ \overline{\xi_2\xi_1} & \overline{\xi_2^2} & \cdots & \overline{\xi_2\xi_n} \\ \vdots & \vdots & & \vdots \\ \overline{\xi_n\xi_1} & \overline{\xi_n\xi_2} & \cdots & \overline{\xi_n^2} \end{pmatrix} \tag{2·1·2}$$

and

$$\Delta = |\,\Xi_{ij}\,|. \tag{2·1·3}$$

The elements of (Ξ_{ij}) are the mean products $\overline{\xi_i\xi_j}$ of the variables ξ_i and ξ_j over the probability space of the independent components. (Ξ_{ij}) is a positive-definite matrix, for if $\alpha_1, \ldots, \alpha_n$ are any n parameters not all zero

$$\alpha_i\alpha_j\overline{\xi_i\xi_j} = \overline{(\alpha_i\xi_i)^2} > 0. \tag{2·1·4}$$

Now according to equation (1·2·1), ζ and also its derivatives are variables of this type. Further, writing for brevity

$$u_n x + v_n y + \sigma_n t + \epsilon_n = \phi_n, \tag{2·1·5}$$

we have

$$\overline{\zeta^2} = \overline{(\sum_n c_n \cos\phi_n)^2} = \sum_n \tfrac{1}{2}c_n^2, \tag{2·1·6}$$

since the phases are random. Thus

$$\overline{\zeta^2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u,v)\,\mathrm{d}u\,\mathrm{d}v = m_{00}, \tag{2·1·7}$$

and accordingly the probability distribution of ξ_1, $= \zeta$, is

$$p(\xi_1) = \frac{1}{(2\pi)^{\frac{1}{2}}m_{00}^{\frac{1}{2}}} \exp\{-\xi_1^2/2m_{00}\}. \tag{2·1·8}$$

Similarly

$$\left.\begin{aligned} \overline{\left(\frac{\partial\zeta}{\partial x}\right)^2} &= \overline{(-\sum_n c_n u_n \sin\phi_n)^2} = \sum_n \tfrac{1}{2}c_n^2 u_n^2 = m_{20}, \\ \overline{\left(\frac{\partial\zeta}{\partial y}\right)^2} &= \overline{(-\sum_n c_n v_n \sin\phi_n)^2} = \sum_n \tfrac{1}{2}c_n^2 v_n^2 = m_{02}, \\ \overline{\frac{\partial\zeta}{\partial x}\frac{\partial\zeta}{\partial y}} &= \overline{(\sum_n c_n u_n \sin\phi_n)\,(\sum_n c_n v_n \sin\phi_n)} = \sum_n \tfrac{1}{2}c_n^2 u_n v_n = m_{11}. \end{aligned}\right\} \tag{2·1·9}$$

The matrix of correlations for

$$\xi_2, \xi_3 = \frac{\partial\zeta}{\partial x}, \frac{\partial\zeta}{\partial y} \tag{2·1·10}$$

is therefore

$$(\Xi_{ij}) = \begin{pmatrix} m_{20} & m_{11} \\ m_{11} & m_{02} \end{pmatrix}, \tag{2·1·11}$$

and the joint-probability distribution is

$$p(\xi_2, \xi_3) = \frac{1}{2\pi\Delta_2^{\frac{1}{2}}} \exp\{-(m_{02}\xi_2^2 - 2m_{11}\xi_2\xi_3 + m_{20}\xi_3^2)/2\Delta_2\}, \tag{2·1·12}$$

where
$$\Delta_2 = \begin{vmatrix} m_{20} & m_{11} \\ m_{11} & m_{02} \end{vmatrix}. \tag{2·1·13}$$

The cross-correlations between ζ and $\partial\zeta/\partial x$, $\partial\zeta/\partial y$ are given by

$$\left.\begin{aligned} \overline{\zeta\frac{\partial\zeta}{\partial x}} &= \overline{(\sum_n c_n\cos\phi_n)\,(-\sum_n c_n u_n\sin\phi_n)} = 0, \\ \overline{\zeta\frac{\partial\zeta}{\partial y}} &= \overline{(\sum_n c_n\cos\phi_n)\,(-\sum_n c_n v_n\sin\phi_n)} = 0, \end{aligned}\right\} \tag{2·1·14}$$

so that ζ and $\partial\zeta/\partial x$, $\partial\zeta/\partial y$ are uncorrelated. The joint-probability distribution of

$$(\xi_1, \xi_2, \xi_3) = (\zeta, \partial\zeta/\partial x, \partial\zeta/\partial y) \tag{2·1·15}$$

is therefore given by
$$p(\xi_1, \xi_2, \xi_3) = p(\xi_1)\,p(\xi_2, \xi_3), \tag{2·1·16}$$

where $p(\xi_1)$ is given by (2·1·8) and $p(\xi_2, \xi_3)$ by (2·1·12).

In general we find, by repeated differentiation,

$$\left.\begin{aligned} \overline{\left(\frac{\partial^{p+q}\zeta}{\partial x^p\,\partial y^q}\right)^2} &= m_{2p,\,2q} \\ \text{and}\quad \overline{\frac{\partial^{p+q}\zeta}{\partial x^p\,\partial y^q}\,\frac{\partial^{p'+q'}\zeta}{\partial x^{p'}\,\partial y^{q'}}} &= (-1)^{\frac{1}{2}(p+q-p'-q')}\,m_{p+p',\,q+q'} \quad \text{or} \quad 0, \end{aligned}\right\} \tag{2·1·17}$$

according as $(p+q-p'-q')$ is even or odd. For example, the derivatives of order $p+q=n$ are not correlated with those of order $p'+q'=n+1$; but they are correlated, negatively in general, with those of order $n+2$.

Slightly different results apply to derivatives involving the time t. We have from (1·2·1)

$$\frac{\partial\zeta}{\partial t} = -\sum_n c_n\sigma_n\sin\,(u_n x+v_n y+\sigma_n t+\epsilon_n), \tag{2·1·18}$$

and so the energy spectrum of $\partial\zeta/\partial t$ is $\sigma^2E(u,v)$. The correlations between the derivatives of ζ and those of $\partial\zeta/\partial t$ are given in terms of the moments

$$\left.\begin{aligned} m'_{pq} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma E(u,v)\,u^p v^q\,\mathrm{d}u\,\mathrm{d}v, \\ m''_{pq} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma^2 E(u,v)\,u^p v^q\,\mathrm{d}u\,\mathrm{d}v \end{aligned}\right\} \tag{2·1·19}$$

of the functions $\sigma E(u,v)$ and $\sigma^2E(u,v)$. As in (2·1·9) we have

$$\overline{\left(\frac{\partial\zeta}{\partial t}\right)^2} = m''_{00}, \quad \overline{\frac{\partial\zeta}{\partial t}\frac{\partial\zeta}{\partial x}} = m'_{10}, \quad \overline{\frac{\partial\zeta}{\partial t}\frac{\partial\zeta}{\partial y}} = m'_{01}, \tag{2·1·20}$$

and in general
$$\overline{\left(\frac{\partial^{p+q+1}\zeta}{\partial x^p\,\partial y^q\,\partial t}\right)^2} = m''_{2p,\,2q} \tag{2·1·21}$$

and
$$\overline{\left(\frac{\partial^{p+q+1}\zeta}{\partial x^p\,\partial y^q\,\partial t}\right)\left(\frac{\partial^{p'+q'}\zeta}{\partial x^{p'}\,\partial y^{q'}}\right)} = (-1)^{\frac{1}{2}(p+q-p'-q'+1)}\,m'_{p+p',\,q+q'} \quad \text{or} \quad 0, \tag{2·1·22}$$

according as $(p+q-p'-q')$ is odd or even. Thus all the correlations of the spatial derivatives of $\partial\zeta/\partial t$ with the spatial derivatives of ζ are expressible in terms of the odd moments of σE. The odd derivatives of $\partial\zeta/\partial t$ are all independent of the odd derivatives of ζ but are correlated with all the even derivatives; and vice versa.

Let us now consider more closely the pattern of surface slopes. If the magnitude of the surface slope is α and its direction is θ we have

$$(\xi_2, \xi_3) = \left(\frac{\partial\zeta}{\partial x}, \frac{\partial\zeta}{\partial y}\right) = (\alpha\cos\theta, \alpha\sin\theta), \tag{2·1·23}$$

and so

$$p(\alpha, \theta) = \left|\frac{\partial(\xi_2, \xi_3)}{\partial(\alpha, \theta)}\right| p(\xi_2, \xi_3) = \alpha p(\xi_2, \xi_3), \tag{2·1·24}$$

or from (2·1·12)

$$p(\alpha, \theta) = \frac{\alpha}{2\pi\Delta_2^{\frac{1}{2}}} \exp\{-\alpha^2(m_{02}\cos^2\theta - 2m_{11}\cos\theta\sin\theta + m_{20}\sin^2\theta)/2\Delta_2\}. \tag{2·1·25}$$

If we take the x axis along the principal direction, so that m_{11} vanishes and $m_{20} \geqslant m_{02}$, then

$$p(\alpha, \theta) = \frac{\alpha^2}{2\pi\Delta_2^{\frac{1}{2}}} \exp\{-\alpha^2(m_{02}\cos^2\theta + m_{20}\sin^2\theta)/2\Delta_2\}. \tag{2·1·26}$$

For a fixed value of θ, the r.m.s. slope is given by

$$\left[\frac{\int_0^\infty \alpha^2 p(\alpha, \theta)\,\mathrm{d}\alpha}{\int_0^\infty p(\alpha, \theta)\,\mathrm{d}\alpha}\right]^{\frac{1}{2}} = \left[\frac{2\Delta_2}{m_{02}\cos^2\theta + m_{20}\sin^2\theta}\right]^{\frac{1}{2}}. \tag{2·1·27}$$

The maximum r.m.s. slope, therefore, is in a direction $\theta = 0$, that is to say, in the principal direction. The minimum slope is in the direction at right angles to this.

The statistical distribution of the slope regardless of direction may be found by integrating $p(\alpha, \theta)$ with respect to θ from 0 to 2π. We find

$$p(\alpha) = \frac{\alpha}{\Delta_2^{\frac{1}{2}}} \exp\{-\alpha^2(m_{20} + m_{02})/4\Delta_2\} I_0[\alpha^2(m_{20} - m_{02})/4\Delta_2], \tag{2·1·28}$$

where

$$I_0(z) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-z\sin\theta}\,\mathrm{d}\theta = J_0(\mathrm{i}z), \tag{2·1·29}$$

I_0 being the Bessel function of order zero with imaginary argument (see Whittaker & Watson 1952, chap. 17). Writing

$$\eta = \frac{\alpha}{(m_{20} + m_{02})^{\frac{1}{2}}} = \frac{\alpha}{m} \tag{2·1·30}$$

for the relative slope, and $\gamma^{-1} = (m_{20}/m_{02})^{\frac{1}{2}}$ for the long-crestedness, we have

$$p(\eta) = \eta(\gamma + \gamma^{-1}) \exp\{-\eta^2(\gamma + \gamma^{-1})^2/4\} I_0[\eta^2(\gamma^{-2} - \gamma^2)/4]. \tag{2·1·31}$$

This distribution is shown in figure 3, for $\gamma = 1, \frac{1}{2}, \frac{1}{4}$ and 0. When $\gamma = 1$ we have, since $I_0(0) = 1$,

$$p(\eta) = 2\eta\,\mathrm{e}^{-\eta^2}. \tag{2·1·32}$$

Thus for short-crested waves the slopes have a Rayleigh distribution. Now as z tends to infinity, $I_0(z) \sim (2\pi z)^{-\frac{1}{2}}\mathrm{e}^z$ (Whittaker & Watson 1952, p. 373) and so when γ is small we have, for general values of η,

$$p(\eta) \sim \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}\eta^2}. \tag{2·1·33}$$

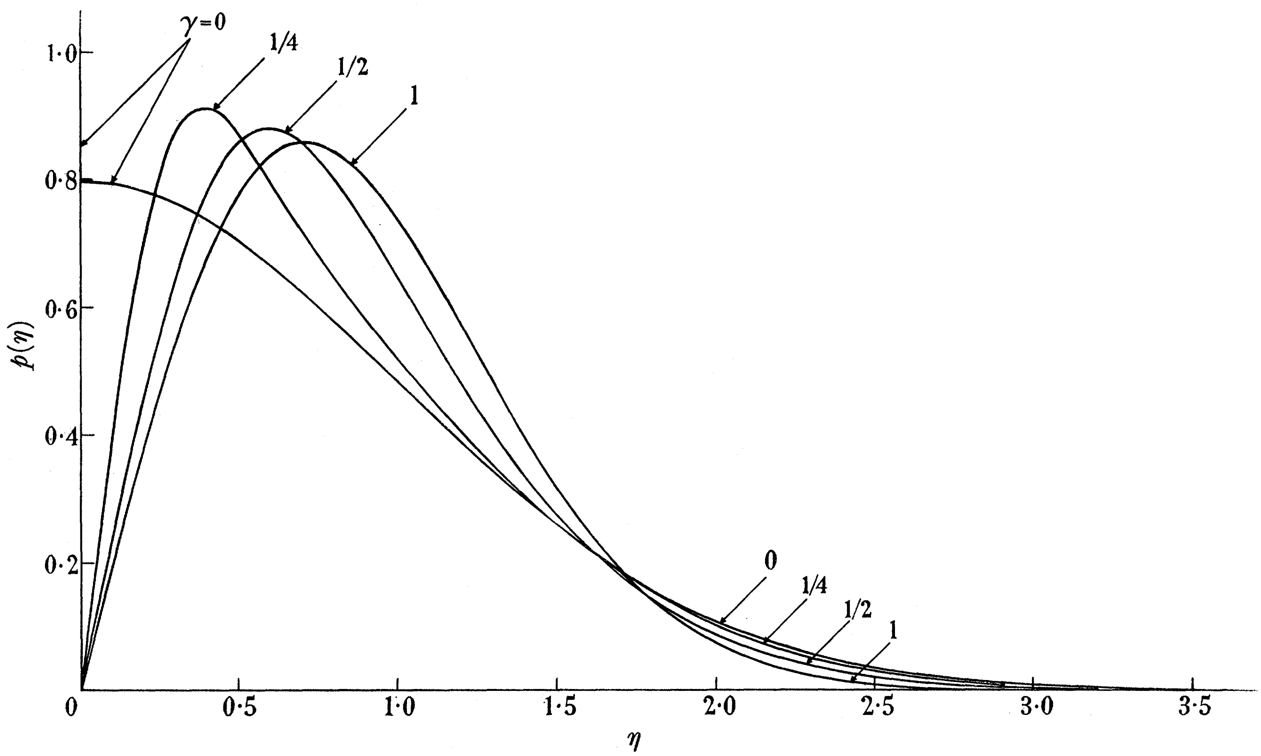

FIGURE 3. The probability distribution of the surface slope $\eta = \alpha/(m_{20}^2+m_{02}^2)^{\frac{1}{2}}$, for different values of the long-crestedness.

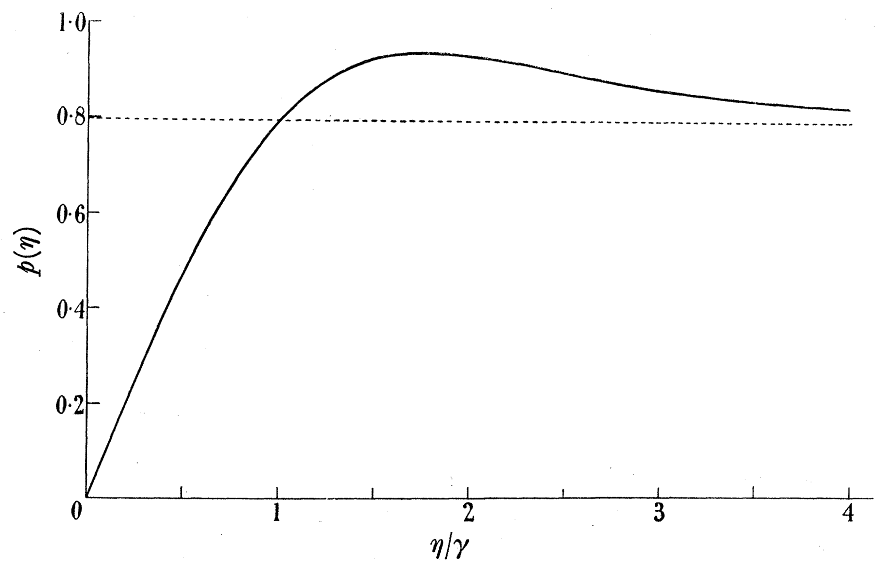

FIGURE 4. The limiting form of the slope distribution close to the origin, for a very long-crested surface.

In other words, for long-crested waves the slopes have in general a normal distribution (as we should expect, for since the slopes are nearly all in one plane, the distribution of α is the same as the distribution of $\partial\zeta/\partial x$, which is normal). However, for very small slopes, comparable with γm, we must use the approximation

$$p(\eta) = (\eta/\gamma)\,\mathrm{e}^{-\eta^2/4\gamma^2}\,I_0(\eta^2/4\gamma^2), \quad = f(\eta/\gamma), \tag{2·1·34}$$

say. $f(\eta/\gamma)$ is plotted in figure 4. As $\eta/\gamma \to \infty$, so $f \to (2/\pi)^{\frac{1}{2}}$, which is the value of (2·1·33) at the origin. The anomalous distribution near the origin appears to arise from directions θ which are nearly perpendicular to the principal direction; since the crests are only of finite length, the chance of a very small slope in this direction is less than if the waves were two-dimensional. Nevertheless, the integral of (2·1·34) from 0 to ∞ is equal to 1, so that as the waves become infinitely long-crested the contribution to the integrated probability from the anomalous term is vanishingly small.

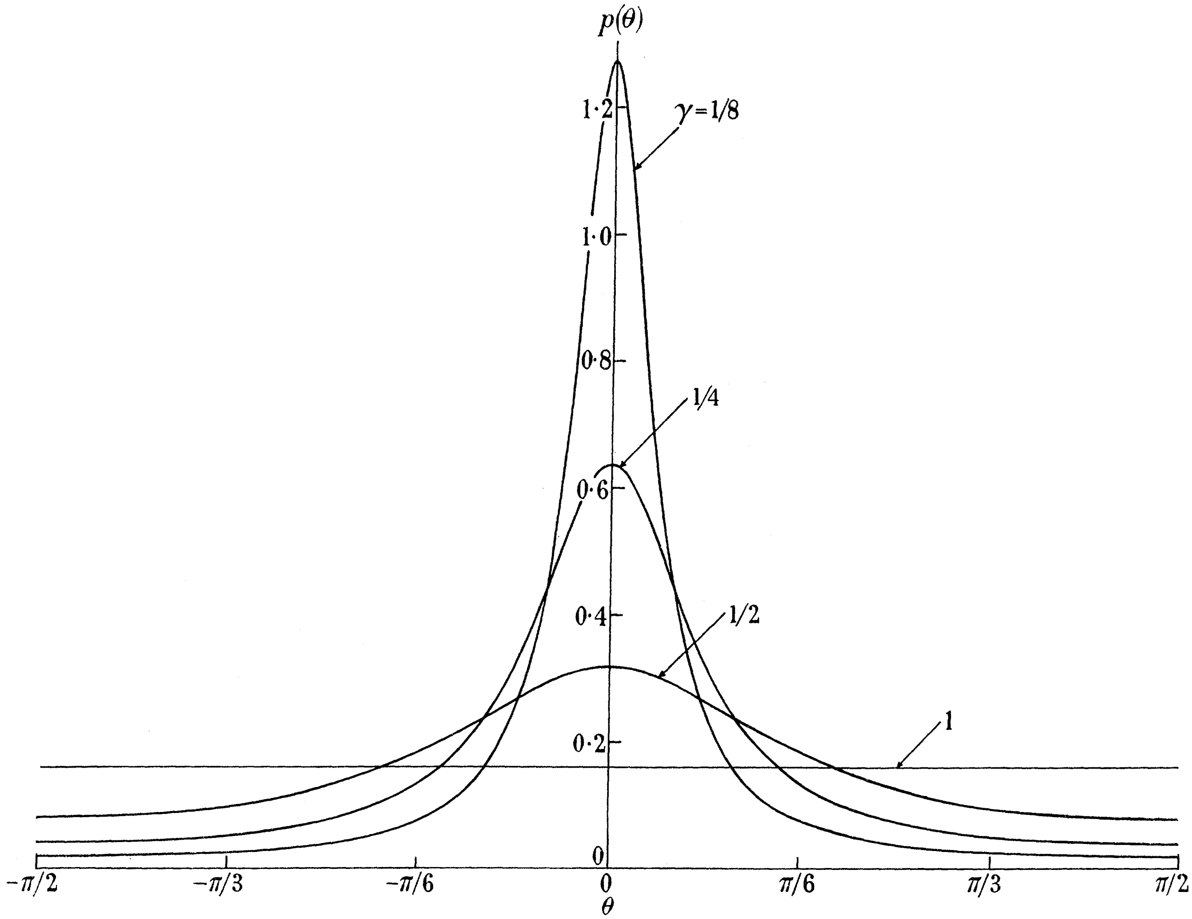

FIGURE 5. The probability distribution of the direction θ of the surface gradient for different values of the long-crestedness. $\theta = 0$ is the principal direction.

Even when the waves are not long-crested, still for large values of η

$$p(\eta) \sim \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{(1-\gamma^2)^{\frac{1}{2}}} e^{-\frac{1}{2}\eta^2}. \tag{2·1·35}$$

Thus for large slopes the distribution always approaches a normal distribution ultimately, provided $\gamma < 1$.

The statistical distribution of the direction θ of the gradient is found by integrating (2·1·26) with respect to α from 0 to ∞:

$$p(\theta) = \frac{\Delta_2^{\frac{1}{2}}}{2\pi(m_{02}\cos^2\theta + m_{20}\sin^2\theta)}. \tag{2·1·36}$$

or

$$p(\theta) = \frac{\gamma}{2\pi(\gamma^2\cos^2\theta + \sin^2\theta)}. \tag{2·1·37}$$

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When $\gamma = 1$ (the waves are short-crested), $p(\theta)$ is independent of θ and there is no preferential direction for the slopes. As γ diminishes the slopes become more and more concentrated about the principal direction (see figure 5). When $\gamma \ll 1$ we have in general

$$p(\theta) = \frac{\gamma}{2\pi \sin^2 \theta},$$

which tends to zero as $\gamma \to 0$. But near the principal direction, i.e. when θ is comparable with γ, we have

$$p(\theta) = \frac{\gamma}{2\pi(\gamma^2+\theta^2)},$$

a distribution whose width is proportional to γ. The integral of the distribution from $\theta/\gamma = -\infty$ to ∞ is equal to $\frac{1}{2}$. Thus the probability that θ is near zero is $\frac{1}{2}$, and so also is the probability that θ is near π. Hence it becomes almost certain that the gradient is nearly in the principal direction.

It should be noted that the probabilities so far discussed are for points distributed randomly and uniformly in the x, y plane. The corresponding probabilities for points selected so as to lie, for example, on a particular contour $\zeta =$ constant, are different, as will be seen in § 2·3.

2·2. *The number of zero-crossings along a line*

As in § 1·4, let us consider the curve in which the surface is intersected by the vertical plane $x \sin\theta = y\cos\theta$. A point where this curve passes through the mean level ($\zeta = 0$) may be called a zero-crossing of ζ. We shall now consider the number of zero-crossings of ζ per unit distance x' measured along the line of intersection of the vertical plane and the mean level.

The mean number of zeros for a random function of a single variable has been derived by Rice (1944, 1945). We recall his argument briefly. ζ and $\partial\zeta/\partial x'$ are random functions which we shall denote by ξ_1 and ξ_2 respectively. Suppose that ζ passes through zero at some point x' in the interval $(x_0', x_0'+dx')$, and with gradient $\partial\zeta/\partial x'$ lying in the range $(\xi_2, \xi_2+d\xi_2)$. Then at the point $x' = x_0$ itself ζ lies in the range $(0, -\xi_2 dx')$, approximately, i.e. a range of height $d\xi_1 = |\xi_2|\, dx'$. The probability of this occurrence is

$$p(0, \xi_2)\, |\xi_2|\, dx'\, d\xi_2, \tag{2·2·1}$$

where $p(\xi_1, \xi_2)$ is the joint-probability distribution of (ξ_1, ξ_2). The total probability of a zero in $(x_0', x_0'+dx')$ is found by integrating with respect to ξ_2 from $-\infty$ to ∞. Hence the total number N_0 of zeros per unit distance is given by

$$N_0 = \int_{-\infty}^{\infty} p(0, \xi_2)\, |\xi_2|\, d\xi_2. \tag{2·2·2}$$

Now the matrix of correlations for (ξ_1, ξ_2) is

$$(\Xi_{ij}) = \begin{pmatrix} m_0 & 0 \\ 0 & m_2 \end{pmatrix}, \tag{2·2·3}$$

and so by (2·1·1)

$$p(\xi_1, \xi_2) = \frac{1}{2\pi(m_0 m_2)^{\frac{1}{2}}} \exp\{-\xi_1^2/2m_0 - \xi_2^2/2m_2\}. \tag{2·2·4}$$

On substituting in (2·2·2) and carrying out the integration we find

$$N_0 = \frac{1}{\pi}\left(\frac{m_2(\theta)}{m_0(\theta)}\right)^{\frac{1}{2}}. \qquad (2\cdot2\cdot5)$$

In other words πN_0 is equal to the r.m.s. wave-number in the direction θ. It follows at once from § 1·4 that

(1) *the number of zeros is a maximum and a minimum for two directions at right angles, given by*

$$\tan 2\theta_p = \frac{2m_{11}}{m_{20}-m_{02}}; \qquad (2\cdot2\cdot6)$$

(2) *the maximum and minimum values of* N_0 *are given by*

$$N_{0\,\text{max.}}, N_{0\,\text{min.}} = \frac{1}{\pi m_{00}^{\frac{1}{2}}}[(m_{20}+m_{02}) \pm \surd\{(m_{20}-m_{02})^2+4m_{11}^2\}]^{\frac{1}{2}}; \qquad (2\cdot2\cdot7)$$

(3) *the number of zeros in a general direction* θ *is given by*

$$N_0^2 = N_{0\,\text{max.}}^2 \cos^2(\theta-\theta_p) + N_{0\,\text{min.}}^2 \sin^2(\theta-\theta_p); \qquad (2\cdot2\cdot8)$$

(4) *the ratio* $N_{0\,\text{min.}}/N_{0\,\text{max.}}$ *is given by*

$$\frac{N_{0\,\text{min.}}}{N_{0\,\text{max.}}} = \gamma, \qquad (2\cdot2\cdot9)$$

where γ^{-1} is the long-crestedness; for a narrow spectrum $N_{0\,\text{min.}}/N_{0\,\text{max.}}$ equals the r.m.s. angular deviation of the energy from the mean direction.

There are similar relations for the mean number of crests and troughs along the curve, since these are simply zeros of the derivative $\partial\zeta/\partial x'$. The energy spectrum of $\partial\zeta/\partial x'$ is u'^2 times the energy spectrum of ζ. So the mean number N_1 of crests and troughs together is

$$N_1 = \frac{1}{\pi}\left(\frac{m_4(\theta)}{m_2(\theta)}\right)^{\frac{1}{2}} \qquad (2\cdot2\cdot10)$$

(the number of crests or troughs separately is half this). m_2 and m_4 can be expressed in terms of the two-dimensional moments of $E(u,v)$ by means of (1·4·12). m_4 is of the fourth degree in $\cos\theta$ and $\sin\theta$, and it is found that in general $N_1(\theta)$ has four maxima (in two pairs of opposite directions) and similarly four minima; these can be found, if necessary, in terms of the fourth-order moments $m_{40}, m_{31}, \ldots, m_{04}$ and the second-order moments m_{20}, m_{11}, m_{02}.

In the same way the mean number N_2 of points of inflexion on the curve is

$$N_2 = \frac{1}{\pi}\left(\frac{m_6(\theta)}{m_4(\theta)}\right), \qquad (2\cdot2\cdot11)$$

and the maxima and minima of N_2 are given in terms of the sixth-, fourth- and second-order moments of $E(u,v)$.

We may find similarly the average number of times that the curve of intersection crosses the level $\zeta = \xi_1$. For in (2·2·1) and (2·2·2) it is necessary only to replace $p(0,\xi_2)$ by $p(\xi_1,\xi_2)$. Since ξ_1 and ξ_2 are independent, this simply amounts to multiplying by a factor $\exp\{-\xi_1^2/2m_0\}$. So in the general case we have

$$N_0 = \frac{1}{\pi}\left(\frac{m_2(\theta)}{m_0(\theta)}\right)^{\frac{1}{2}} \exp\{-\xi_1^2/2m_0(\theta)\}. \qquad (2\cdot2\cdot12)$$

By similar reasoning, the number of times N_1 per unit distance that the curve has a gradient ξ_2 is given by

$$N_1 = \frac{1}{\pi}\left(\frac{m_4(\theta)}{m_2(\theta)}\right)^{\frac{1}{2}} \exp\{-\xi_2^2/2m_2(\theta)\}, \tag{2·2·13}$$

and there are similar expressions corresponding to the higher derivatives of $\zeta(x')$.

2·3. *The length and direction of the contours*

Let us consider now a corresponding property in two dimensions. Imagine the surface contours $\zeta =$ constant to be drawn in the x, y plane. Contained in any region A of the plane there will be a certain length s of the contour $\zeta = \zeta_0$. The average length of contour, being proportional to the area A, may be denoted by $\bar{s}A$. The factor $\bar{s}$ is now to be evaluated.

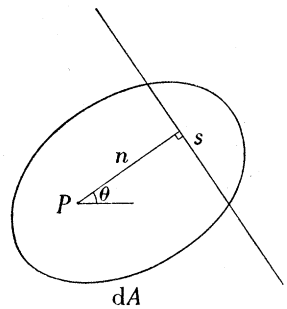


FIGURE 6. The length s of contour intercepted by a small element of area dA.

Let P be any fixed point in the plane, and dA the area of a small region surrounding P (see figure 6). Let s denote the length of a contour $\zeta = \zeta_0$ intercepted by the region dA, and let n denote the perpendicular distance of the contour from P. Suppose that the magnitude α and direction θ of the gradient are fixed. Then the height ζ of the surface at P is given by

$$|\zeta - \zeta_0| = \alpha n. \tag{2·3·1}$$

For the contour to cut the element of area, the perpendicular n and the height ζ must lie in certain small ranges (n_1, n_2) and (ζ_1, ζ_2). If now α, θ are allowed to vary within small ranges $(\alpha, \alpha + \mathrm{d}\alpha)$, $(\theta + \mathrm{d}\theta)$, the expectation $\bar{s}_{\alpha,\theta}\mathrm{d}A\,\mathrm{d}\alpha\,\mathrm{d}\theta$ of s over the area dA is given by

$$\bar{s}_{\alpha,\theta}\mathrm{d}A\,\mathrm{d}\alpha\,\mathrm{d}\theta = \int_{\zeta_1}^{\zeta_2} sp(\zeta,\alpha,\theta)\,\mathrm{d}\zeta\,\mathrm{d}\alpha\,\mathrm{d}\theta = \int_{n_1}^{n_2} sp(\zeta,\alpha,\theta)\,\alpha\,\mathrm{d}n\,\mathrm{d}\alpha\,\mathrm{d}\theta, \tag{2·3·2}$$

where $p(\zeta, \alpha, \theta)$ denotes the joint distribution of ζ, α, θ at P. Since ζ, α, θ are nearly constant over the small range of integration of n we have

$$\bar{s}_{\alpha,\theta}\mathrm{d}A = \alpha p(\zeta,\alpha,\theta)\int_{n_1}^{n_2} s\,\mathrm{d}n = \alpha p(\zeta,\alpha,\theta)\,\mathrm{d}A, \tag{2·3·3}$$

and so

$$\bar{s}_{\alpha,\theta} = \alpha p(\zeta,\alpha,\theta). \tag{2·3·4}$$

Integrating over all possible values of α, θ we have

$$\bar{s} = \int_0^\infty \int_0^{2\pi} \bar{s}_{\alpha,\theta}\,\mathrm{d}\alpha\,\mathrm{d}\theta = \int_0^\infty \int_0^{2\pi} \alpha p(\zeta,\alpha,\theta)\,\mathrm{d}\alpha\,\mathrm{d}\theta. \tag{2·3·5}$$

Now from § 2·1

$$p(\zeta,\alpha,\theta) = p(\zeta)\,p(\alpha,\theta), \tag{2·3·6}$$

where $p(\zeta)$ is given by (2·1·8), with $\xi_1 = \zeta$, and $p(\alpha, \theta)$ is given by (2·1·20). On substituting these values in (2·3·5) we have

$$\bar{s} = \frac{1}{(2\pi)^{\frac{3}{2}}(m_{00}\Delta_2)^{\frac{1}{2}}} \exp\{-\zeta^2/2m_{00}\} \int_0^\infty \int_0^{2\pi} \alpha^2 \exp\{-\alpha^2(m_{02}\cos^2\theta + m_{20}\sin^2\theta)/2\Delta_2\}\,d\alpha\,d\theta. \tag{2·3·7}$$

(It has been supposed that the x axis is taken in the principal direction, so that m_{11} vanishes.) Integration with respect to α gives

$$\bar{s} = \frac{\Delta_2}{4\pi m_{00}^{\frac{1}{2}}} \exp\{-\xi^2/2m_{00}\} \int_0^{2\pi} \frac{d\phi}{(m_{02}\sin^2\phi + m_{20}\cos^2\phi)^{\frac{3}{2}}}, \tag{2·3·8}$$

where $\phi = \theta + \frac{1}{2}\pi$. That is to say

$$\bar{s} = \frac{m_{02}}{\pi(m_{00}m_{20})^{\frac{1}{2}}} \exp\{-\zeta^2/2m_{00}\} \int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1-k^2\sin^2\phi)^{\frac{3}{2}}}, \tag{2·3·9}$$

where
$$k^2 = 1 - \gamma^2. \tag{2·3·10}$$

Now since
$$\frac{d}{d\phi}\frac{k^2\sin\phi\cos\phi}{(1-k^2\sin^2\phi)^{\frac{1}{2}}} = (1-k^2\sin^2\phi)^{\frac{1}{2}} + \frac{k^2-1}{(1-k^2\sin^2\phi)^{\frac{3}{2}}}, \tag{2·3·11}$$

it follows, on integration between 0 and $\frac{1}{2}\pi$, that

$$(1-k^2)\int_0^{\frac{1}{2}\pi} \frac{d\phi}{(1-k^2\sin^2\phi)^{\frac{3}{2}}} = \int_0^{\frac{1}{2}\pi} (1-k^2\sin^2\phi)^{\frac{1}{2}}\,d\phi = E(k), \tag{2·3·12}$$

where $E(k)$ is Legendre's complete elliptic integral of the first kind (Legendre 1811). Hence we have finally

$$\bar{s} = \frac{1}{\pi}\left(\frac{m_{20}+m_{02}}{m_{00}}\right)^{\frac{1}{2}} \exp\{-\zeta^2/2m_{00}\}\,(1+\gamma^2)^{-\frac{1}{2}}\,E\{\surd(1-\gamma^2)\}. \tag{2·3·13}$$

In the special case of long-crested waves, when $\gamma = 0$, we have $E(1) = 1$ and further

$$m_{20} = m_2(0), \quad m_{02} = 0, \tag{2·3·14}$$

giving
$$\bar{s} = \frac{1}{\pi}\left(\frac{m_2(0)}{m_{00}}\right)^{\frac{1}{2}} \exp\{-\zeta^2/2m_{00}\}. \tag{2·3·15}$$

Comparison with (2·2·12) shows what we might expect, namely, that the mean length of contour per unit area is equal to the mean number of crossings of the contour level per unit distance by a plane perpendicular to the wave crests.

In general (2·3·13) may be written

$$\bar{s} = \frac{1}{\pi}\left(\frac{m_{20}+m_{02}}{m_{00}}\right)^{\frac{1}{2}} \exp\{-\zeta^2/2m_{00}\} f(\gamma), \tag{2·3·16}$$

where
$$f(\gamma) = (1+\gamma^2)^{-\frac{1}{2}}\,E\{\surd(1-\gamma^2)\}. \tag{2·3·17}$$

This function is shown in figure 7. At the two extreme values we have

$$f(0) = 1 \tag{2·3·18}$$

and
$$f(1) = \frac{\pi}{2\surd 2} = 1{\cdot}1107\ldots. \tag{2·3·19}$$

Throughout its whole range the function departs very little from unity. There is, however, a weak singularity at the origin, where

$$f(\gamma) = 1+\tfrac{1}{2}\gamma^2\left(\ln\frac{4}{\gamma}-\frac{3}{2}\right)+O(\gamma^2\ln\gamma). \tag{2·3·20}$$

A very closely related distribution is that of the direction θ of the normal to a given contour. Let us suppose that θ is measured at points randomly and uniformly distributed along the contour. θ is also the direction of the surface gradient at the point of measurement. However, the distribution of θ for a given contour is quite distinct from the distribution of θ found in § 2·1, where it was supposed that the angle was measured, not on a particular contour but at points randomly distributed in the x, y plane.

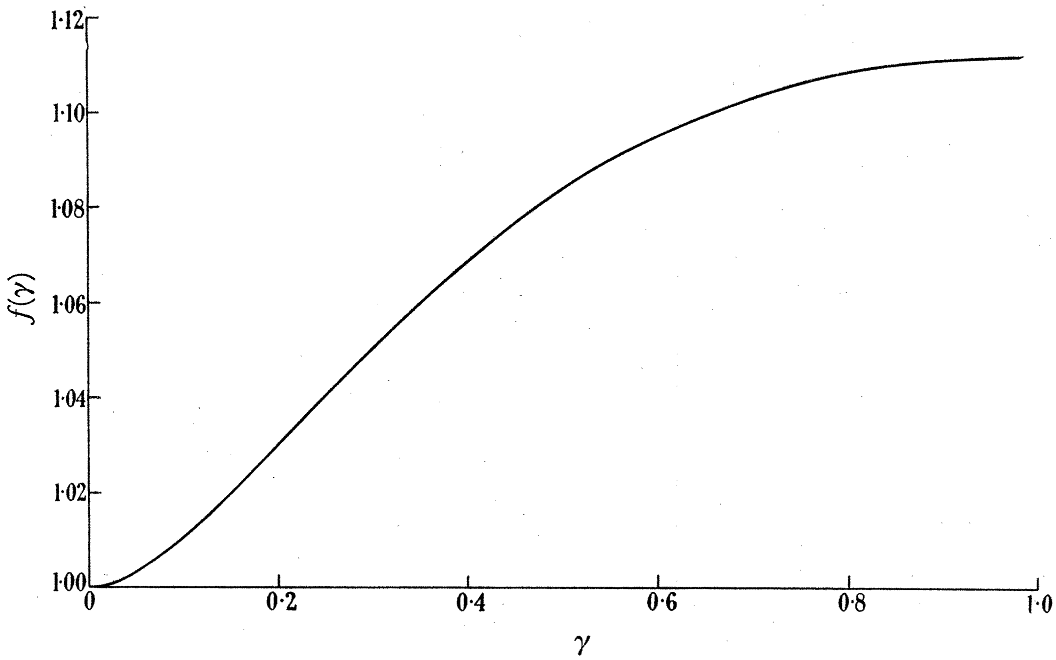


FIGURE 7. Graph of $f(\gamma) = (1+\gamma^2)^{-\frac{1}{2}} E[\surd(1-\gamma^2)\}$.

To find the distribution $p(\theta)_\zeta$ for the contour $\zeta =$ constant we may note that the contribution of a given length of arc to the distribution of θ in the interval $(\theta, \theta+\mathrm{d}\theta)$ is simply proportional to the expected length of arc for which θ lies between these limits, that is,

$$p(\theta)_\zeta \,\mathrm{d}\theta \propto \int_0^\infty \bar{s}_{\alpha,\theta}\,\mathrm{d}\alpha\,\mathrm{d}\theta. \tag{2·3·21}$$

On normalizing the right-hand side by dividing by $\bar{s}$ we have

$$p(\theta)_\zeta = \frac{1}{\bar{s}}\int_0^\infty \bar{s}_{\alpha,\theta}\,\mathrm{d}\alpha = \frac{1}{\bar{s}}\int_0^\infty \alpha p(\zeta,\alpha,\theta)\,\mathrm{d}\alpha. \tag{2·3·22}$$

Substituting from (2·3·6) and (2·3·13), and carrying out the integration we find

$$p(\theta)_\zeta = \frac{1}{4\mathrm{E}\{\surd(1-\gamma^2)\}}\frac{\gamma^2}{(\gamma^2\cos^2\theta+\sin^2\theta)^{\frac{3}{2}}} \tag{2·3·23}$$

(where $\theta = 0$ is chosen as the principal direction). The form of this expression is somewhat similar to (2·1·37). When $\gamma = 1$ (for short-crested waves)

$$p(\theta)_\zeta = \frac{1}{2\pi}, \tag{2·3·24}$$

i.e. the contours have no preferential direction. As γ diminishes the distribution becomes more and more concentrated about the mean direction $\theta = 0$. When γ is small, we have for general directions

$$p(\theta)_\zeta = \frac{\gamma^2}{4\sin^3\theta}, \tag{2·3·25}$$

which tends to zero as $\gamma \to 0$. But near the mean direction, that is, when θ is comparable with γ,

$$p(\theta)_\zeta = \frac{\gamma^2}{4(\gamma^2+\theta^2)^{\frac{3}{2}}}. \tag{2·3·26}$$

The integral of this expression from $\theta/\gamma = -\infty$ to ∞ is equal to $\frac{1}{2}$. Thus the probability that θ is near zero is $\frac{1}{2}$, and so also is the probability that θ is near π. Hence it becomes highly probable that the direction of the contour is near the principal direction.

2·4. *The density of maxima and minima*

Let us consider now the problem of how many maxima and minima (humps and hollows) the surface possesses, on the average, per unit area.

At a maximum or a minimum the two components of gradient $\partial\zeta/\partial x$, $\partial\zeta/\partial y$ must vanish. But not all such points are maxima or minima; we may also have a col or saddle-point, where the surface tends to rise in one pair of opposite directions and fall in another pair of opposite directions. We shall prove the following theorem:

On a statistically uniform surface the average density of maxima per unit area plus the average density of minima is equal to the average density of saddle-points, or

$$D_{\text{ma.}} + D_{\text{mi.}} = D_{\text{sa.}}. \tag{2·4·1}$$

Let a contour map of the surface be drawn, and let a direction ϕ be assigned to each contour, say to the right when facing up-hill. Thus at each point of the plane, except the stationary points, there is a unique direction ϕ. Consider now the variation of ϕ round a small closed curve C on the map (see figure 8). C may at first be so small as to contain no stationary point, in which case ϕ will return to its initial value after the circuit is completed (figure 8 *a*). If now C is expanded so as to enclose a single stationary point, ϕ will increase by 2π on completion of the circuit C if the stationary point is a maximum or a minimum (figure 8 *b* and *c*), and will decrease by 2π if the stationary point is a saddle-point (figure 8 *d*). As C is further increased in size, so as to enclose $d_{\text{ma.}}$ maxima, $d_{\text{mi.}}$ minima and $d_{\text{sa.}}$ saddle-points, say, the variation of ϕ round C will be $2\pi(d_{\text{ma.}}+d_{\text{mi.}}-d_{\text{sa.}})$, or $2\pi A(D_{\text{ma.}}+D_{\text{mi.}}-D_{\text{sa.}})$ approximately, where A is the area enclosed by C. But since the surface is statistically uniform, the variation of ϕ round C will increase proportionally to L at most, where L is the circumference of C.* On the other hand A increases like L^2, supposing C is of constant shape. Thus $2\pi(D_{\text{ma.}}+D_{\text{mi.}}-D_{\text{sa.}})$ is proportional to L^{-1} at most, and letting L tend to infinity we see that $(D_{\text{ma.}}+D_{\text{mi.}}-D_{\text{sa.}})$ must vanish. This proves the result.

* In fact it may be shown that the increase is proportional only to $L^{\frac{1}{2}}$.

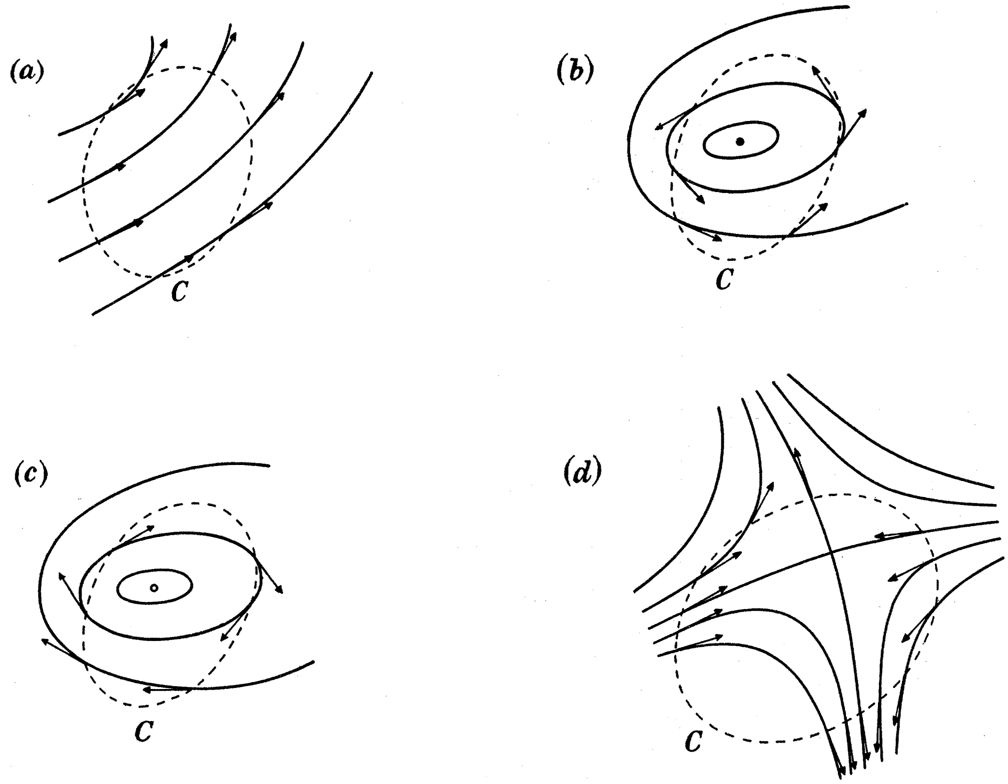


FIGURE 8. Illustrating the way in which the contour direction varies round a curve enclosing (*a*) no stationary point, (*b*) a maximum, (*c*) a minimum and (*d*) a saddle-point.

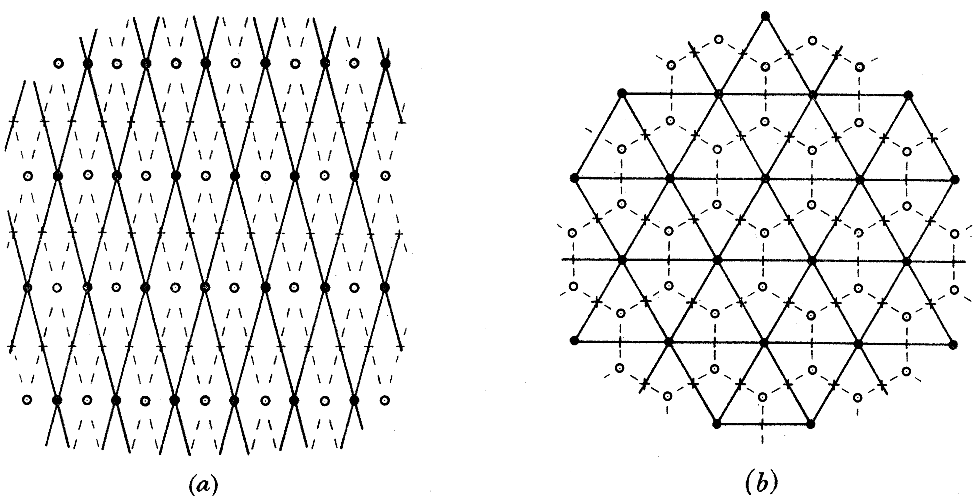


FIGURE 9. (*a*) Stationary points on a surface which consists of two intersecting wave systems. ● = a maximum, ○ = a minimum, + = a saddle-point. (*b*) Stationary points in a hexagonal pattern.

A simple example is shown in figure 9 *a*. The surface consists of two long-crested systems of waves of slowly varying amplitude. Where a crest from one system intersects a crest from the other system there is a maximum, and where two troughs intersect there is a minimum. But where a crest from one system intersects a trough from theother there is a saddle-point.

The pattern of saddle-points is similar and congruent to the pattern of maxima and minima together, so that (2·4·1) is satisfied.

On this surface the density of maxima is equal to the density of minima. But a case in which this is not so is illustrated in figure 9 *b*. Here the maxima are at the centres of the cells of a hexagonal honeycomb, the minima are at the vertices and the saddle-points are half-way along the edges. There are twice as many maxima as minima, and three times as many saddle-points, so that

$$D_{\text{ma.}} = \tfrac{2}{3}D_{\text{sa.}}, \quad D_{\text{mi.}} = \tfrac{1}{3}D_{\text{sa.}}. \tag{2·4·2}$$

In general it can be shown that the stationary points must form a cellular pattern, and the theorem (2·4·1) then follows from Euler's relation $V+F = E+2$ connecting the number of vertices V, faces F and edges E of a convex polyhedron (Euler 1752–3; Sommerville 1929, chap. IX).

The class of random surface represented by (1·2·1) satisfies the further relation

$$D_{\text{ma.}} = D_{\text{mi.}}. \tag{2·4·3}$$

For the phases ϵ_n of the component waves are randomly and uniformly distributed between 0 and 2π. The statistical properties of the surface are unaffected if a constant, π, is added to each phase. But this reverses the sign of ζ and converts maxima into minima, and vice versa.

From (2·4·1) and (2·4·3) it follows that

$$D_{\text{sa.}} = 2D_{\text{ma.}} = 2D_{\text{mi.}} \tag{2·4·4}$$

and if $D_{\text{sta.}}$ denotes the total density of stationary points per unit area of the surface

$$D_{\text{sta.}} = 2D_{\text{sa.}} = 4D_{\text{ma.}} = 4D_{\text{mi.}}. \tag{2·4·5}$$

In other words, of all the stationary points on the surface, one-quarter are maxima, one-quarter are minima, and the remaining half are saddle-points.

We proceed now to evaluate $D_{\text{sta.}}$ in terms of the energy spectrum of ζ. The variables entering the problem are

$$\frac{\partial\zeta}{\partial x}, \frac{\partial\zeta}{\partial y} = \xi_2, \xi_3 \tag{2·4·6}$$

and

$$\frac{\partial^2\zeta}{\partial x^2}, \frac{\partial^2\zeta}{\partial x\partial y}, \frac{\partial^2\zeta}{\partial y^2} = \xi_4, \xi_5, \xi_6, \tag{2·4·7}$$

say. (ξ_2, ξ_3) is a pair of functions of (x, y), and if (x, y) varies within a certain small region $\mathrm{d}A$, $= (x, x+\mathrm{d}x; y, y+\mathrm{d}y)$, (ξ_2, ξ_3) will vary within a region $\mathrm{d}\Sigma$ of area

$$|\mathrm{d}\Sigma| = |J|\,|\mathrm{d}A|, \tag{2·4·8}$$

where

$$J = \frac{\partial(\xi_2, \xi_3)}{\partial(x, y)} = \xi_4\xi_6 - \xi_5^2. \tag{2·4·9}$$

The probability that a given point, say a stationary point, lies in $\mathrm{d}A$ is equal to the probability that (ξ_2, ξ_3) lies in the corresponding region $\mathrm{d}\Sigma$, which is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\xi_4\,\mathrm{d}\xi_5\,\mathrm{d}\xi_6\iint_{\mathrm{d}\Sigma}\mathrm{d}\xi_2\,\mathrm{d}\xi_3\,p(\xi_2, \xi_3, \xi_4, \xi_5, \xi_6). \tag{2·4·10}$$

Since $(\xi_2, \xi_3) = (0, 0)$ somewhere in $\mathrm{d}\Sigma$, $p(\xi_2, \xi_3, \xi_4, \xi_5, \xi_6)$ may be replaced by

$$p(0, 0, \xi_4, \xi_5, \xi_6)$$

when $d\Sigma$ is sufficiently small, and since

$$\iint_{d\Sigma} d\xi_2 d\xi_3 = |d\Sigma|, \tag{2·4·11}$$

the above probability becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\xi_4 d\xi_5 d\xi_6 \,|d\Sigma|\, p(0,0,\xi_4,\xi_5,\xi_6). \tag{2·4·12}$$

On substituting from (2·4·8) we have for the probability of a stationary value of ζ in dA,

$$D_{\text{sta.}}\, dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(0,0,\xi_4,\xi_5,\xi_6)\,|\xi_4\xi_6-\xi_5^2|\, d\xi_4 d\xi_5 d\xi_6 dA. \tag{2·4·13}$$

For a true maximum of the surface we must have $\xi_4 \leqslant 0$, $\xi \leqslant_6 0$ and $J \geqslant 0$; for a true minimum, $\xi_4 \geqslant 0$, $\xi_6 \geqslant 0$ and $J \geqslant 0$. Thus the true maxima and minima correspond to the region of the (ξ_4, ξ_5, ξ_6) space given by

$$J \equiv \xi_4\xi_6 - \xi_5^2 \geqslant 0. \tag{2·4·14}$$

The boundary of this region is the surface $J = 0$, which is a cone with vertex at the origin. The remaining part of the (ξ_4, ξ_5, ξ_6) space corresponds to the saddle-points.

Now since the second derivatives ξ_4, ξ_5, ξ_6 are uncorrelated with the first derivatives ξ_2, ξ_3 (see § 2·1) it follows that

$$p(\xi_2,\xi_3,\xi_4,\xi_5,\xi_6) = p(\xi_2,\xi_3)\, p(\xi_4,\xi_5,\xi_6), \tag{2·4·15}$$

where $p(\xi_2, \xi_3)$ is given by (2·1·12) and $p(\xi_4, \xi_5, \xi_6)$ is the distribution for (ξ_4, ξ_5, ξ_6) independently of the other variables. The matrix of correlations is

$$(\Xi_{ij}) = \begin{pmatrix} m_{40} & m_{31} & m_{22} \\ m_{31} & m_{22} & m_{13} \\ m_{22} & m_{13} & m_{04} \end{pmatrix}, \tag{2·4·16}$$

and hence

$$p(\xi_4,\xi_5,\xi_6) = \frac{1}{(2\pi)^{\frac{3}{2}}\Delta_4^{\frac{1}{2}}} \exp\{-\tfrac{1}{2}M_{ij}\xi_{i+3}\xi_{j+3}\}, \tag{2·4·17}$$

where (M_{ij}) is the inverse matrix to (Ξ_{ij}) and

$$\Delta_4 = |\Xi_{ij}|. \tag{2·4·18}$$

Therefore, altogether we have for the density of stationary points

$$D_{\text{sta.}} = \frac{1}{(2\pi)^{\frac{5}{2}}\Delta_2^{\frac{1}{2}}\Delta_4^{\frac{1}{2}}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}M_{ij}\xi_{i+3}\xi_{j+3}\}\,|\xi_4\xi_6-\xi_5^2|\, d\xi_4 d\xi_5 d\xi_6. \tag{2·4·19}$$

The density of maxima is given by a similar integral taken over the region $\xi_4 \leqslant 0$, $\xi_6 \leqslant 0$, $J \geqslant 0$. The density of saddle-points is given by the same integral taken over the region $J < 0$.

Since (Ξ_{ij}) is a positive-definite matrix, so also is its inverse (M_{ij}), and there exists a real linear transformation

$$(\xi_4, \xi_5, \xi_6) = T(\eta_1, \eta_2, \eta_3) \tag{2·4·20}$$

which simultaneously reduces the exponent in (2·4·19) to the unit form

$$M_{ij}\xi_{i+3}\xi_{j+3} = \eta_1^2 + \eta_2^2 + \eta_3^2 \tag{2·4·21}$$

and J to a diagonal form

$$\xi_4\xi_6 - \xi_5^2 = l_1\eta_1^2 + l_2\eta_2^2 + l_3\eta_3^2. \tag{2·4·22}$$

The quantities l_1, l_2, l_3 are easily found, for they are the roots of

$$|\sigma_{ij}-lM_{ij}|=0, \tag{2·4·23}$$

where (σ_{ij}) is the matrix of J:

$$(\sigma_{ij})=\begin{pmatrix}0 & 0 & \frac{1}{2}\\ 0 & -1 & 0\\ \frac{1}{2} & 0 & 0\end{pmatrix}. \tag{2·4·24}$$

On multiplying (2·4·23) by $|\Xi_{ij}|$, $=|M_{ij}^{-1}|$, we have

$$|\Xi_{ij}\sigma_{jk}-l\delta_{ik}|=0, \tag{2·4·25}$$

where δ_{ij} is the unit matrix of order 3. In other words l_1, l_2, l_3 are the latent roots of $(\Xi_{ij}\sigma_{jk})$:

$$\begin{vmatrix}\frac{1}{2}m_{22}-l & -m_{31} & \frac{1}{2}m_{40}\\ \frac{1}{2}m_{13} & -m_{22}-l & \frac{1}{2}m_{31}\\ \frac{1}{2}m_{04} & -m_{13} & \frac{1}{2}m_{22}-l\end{vmatrix}=0. \tag{2·4·26}$$

On expanding the determinant we find

$$4l^3-3Hl-\Delta_4=0, \tag{2·4·27}$$

where

$$3H=m_{40}m_{04}-4m_{31}m_{13}+3m_{22}^2. \tag{2·4·28}$$

Hence

$$l_1+l_2+l_3=0 \tag{2·4·29}$$

and

$$l_1l_2l_3=\tfrac{1}{4}\Delta_4>0. \tag{2·4·30}$$

It follows that one of the roots, say l_1, is positive and the other two, say l_2, l_3, are negative. We write

$$l_1>0>l_2\geqslant l_3. \tag{2·4·31}$$

The solution of the cubic equation (2·4·27) is

$$(l_1,l_2,l_3)=H^{\frac{1}{2}}(\cos\psi_1,\cos\psi_2,\cos\psi_3), \tag{2·4·32}$$

where ψ_1, ψ_2, ψ_3 are the roots of

$$\cos 3\psi=\Delta_4/H^{\frac{3}{2}}. \tag{2·4·33}$$

The modulus of the transformation T is

$$\frac{\partial(\xi_4,\xi_5,\xi_6)}{\partial(\eta_1,\eta_2,\eta_3)}=|M_{ij}|^{-\frac{1}{2}}=\Delta_4^{\frac{1}{2}}. \tag{2·4·34}$$

We find then

$$D_{\text{sta.}}=\frac{1}{(2\pi)^{\frac{5}{2}}\Delta_2^{\frac{1}{2}}}I(l_1,l_2,l_3), \tag{2·4·35}$$

where

$$I(l_1,l_2,l_3)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{-\tfrac{1}{2}(\eta_1^2+\eta_2^2+\eta_3^2)\}\,|l_1\eta_1^2+l_2\eta_2^2+l_3\eta_3^2|\,\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\eta_3. \tag{2·4·36}$$

The density of maxima is given by

$$D_{\text{ma.}}=\frac{1}{(2\pi)^{\frac{5}{2}}\Delta_2^{\frac{1}{2}}}I'(l_1,l_2,l_3), \tag{2·4·37}$$

where

$$I'(l_1,l_2,l_3)=\iiint_V\exp\{-\tfrac{1}{2}(\eta_1^2+\eta_2^2+\eta_3^2)\}\,|l_1\eta_1^2+l_2\eta_2^2+l_3\eta_3^2|\,\mathrm{d}\eta_1\mathrm{d}\eta_2\mathrm{d}\eta_3, \tag{2·4·38}$$

and V is the conical region

$$\eta_1 > 0, \quad l_1\eta_1^2 + l_2\eta_2^2 + l_3\eta_3^2 > 0. \tag{2·4·39}$$

Clearly
$$4I' - I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}(\eta_1^2+\eta_2^2+\eta_3^2)\}\,(l_1\eta_1^2+l_2\eta_2^2+l_3\eta_3^2)\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_3$$
$$= (2\pi)^{\frac{3}{2}}(l_1+l_2+l_3), \tag{2·4·40}$$

which vanishes, by (2·4·29). Thus

$$I = 4I', \quad D_{\text{sta.}} = 4D_{\text{ma.}}, \tag{2·4·41}$$

in agreement with (2·4·5).

The integral I' may be evaluated by means of the substitution

$$\left.\begin{aligned} \eta_1 &= l_1^{-\frac{1}{2}} r, \\ \eta_2 &= (-l_2)^{-\frac{1}{2}} r \sin\theta \cos\chi, \\ \eta_3 &= (-l_3)^{-\frac{1}{2}} r \sin\theta \sin\chi, \end{aligned}\right\} \tag{2·4·42}$$

where
$$0 < r < \infty, \quad 0 < \theta < \tfrac{1}{2}\pi, \quad 0 < \chi < 2\pi. \tag{2·4·43}$$

We have then

$$I' = \frac{1}{(l_1 l_2 l_3)^{\frac{1}{2}}} \int_0^{\infty} \mathrm{d}r \int_0^{\frac{1}{2}\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\chi \exp\{-(1+f\sin^2\theta)\,r^2/2l_1\}\, r^4 \cos^3\theta \sin\theta, \tag{2·4·44}$$

where
$$f = f(\chi) = -\frac{l_1}{l_2}\cos^2\chi - \frac{l_1}{l_3}\sin^2\chi. \tag{2·4·45}$$

Integration with respect to r gives

$$I' = 3(\tfrac{1}{2}\pi)^{\frac{1}{2}} \frac{l_1^2}{(l_2 l_3)^{\frac{1}{2}}} \int_0^{\frac{1}{2}\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\chi \frac{\cos^3\theta \sin\theta}{(1+f\sin^2\theta)^{\frac{5}{2}}}. \tag{2·4·46}$$

Further integration with respect to θ gives

$$I' = 4(\tfrac{1}{2}\pi)^{\frac{1}{2}} \frac{l_1^2}{(l_2 l_3)^{\frac{1}{2}}} \int_0^{\frac{1}{2}\pi} \mathrm{d}\chi \left[\frac{1}{f} - \frac{2}{f^2} + \frac{2}{f^2(1+f)^{\frac{1}{2}}}\right]. \tag{2·4·47}$$

This is an elliptic integral and may be evaluated by known methods (Legendre 1811). We find finally

$$I' = (8\pi)^{\frac{1}{2}} \Big[(l_2 l_3)^{\frac{1}{2}} \Big\{ \Big(\frac{l_2-l_1}{l_2}\Big)^{\frac{1}{2}} \mathrm{E}(k, \tfrac{1}{2}\pi) - \Big(\frac{l_2}{l_2-l_1}\Big)^{\frac{1}{2}} \mathrm{F}(k, \tfrac{1}{2}\pi) \Big\}$$
$$- (l_1+l_2+l_3)\{\mathrm{F}(k',\phi)\,\mathrm{E}(k,\tfrac{1}{2}\pi) + \mathrm{E}(k',\phi)\,\mathrm{F}(k,\tfrac{1}{2}\pi) - \mathrm{F}(k',\phi)\,\mathrm{F}(k,\tfrac{1}{2}\pi) - \tfrac{1}{2}\pi\}\Big], \tag{2·4·48}$$

where E and F are the Legendre elliptic integrals of the first and second kind:

$$\left.\begin{aligned} \mathrm{E}(k,\phi) &= \int_0^{\phi} (1-k^2\sin^2\phi)^{\frac{1}{2}}\,\mathrm{d}\phi, \\ \mathrm{F}(k,\phi) &= \int_0^{\phi} (1-k^2\sin^2\phi)^{-\frac{1}{2}}\,\mathrm{d}\phi, \end{aligned}\right\} \tag{2·4·49}$$

and
$$k^2 = \frac{l_1(l_3-l_2)}{l_3(l_1-l_2)}, \qquad k'^2 = 1-k^2 = \frac{l_2(l_1-l_3)}{l_3(l_1-l_2)}, \qquad \phi = \tan^{-1}\Big(-\frac{l^3}{l_1}\Big)^{\frac{1}{2}}. \tag{2·4·50}$$

If we now make use of the condition (2·4·29) we find for the density of maxima

$$D_{\text{ma.}} = \frac{1}{2\pi^2} \frac{(l_2 l_3)^{\frac{1}{2}}}{\Delta_2^{\frac{1}{2}}} \Big[\Big(\frac{l_2-l_1}{l_2}\Big)^{\frac{1}{2}} \mathrm{E}(k, \tfrac{1}{2}\pi) - \Big(\frac{l_2}{l_2-l_1}\Big)^{\frac{1}{2}} \mathrm{F}(k, \tfrac{1}{2}\pi) \Big]. \tag{2·4·51}$$

The density of stationary points $D_{\text{sta.}}$ is four times this value.

Equation (2·4·51) may also be written

$$D_{\text{ma.}} = \frac{1}{2\pi^2}\frac{l_1}{\Delta_2^{\frac{1}{2}}}\Phi(-l_2/l_1), \tag{2·4·52}$$

where

$$\left.\begin{aligned}\Phi(\alpha) &= \{\alpha(1-\alpha)\}^{\frac{1}{2}}\left[\left(\frac{1+\alpha}{\alpha}\right)^{\frac{1}{2}}\mathrm{E}(k,\tfrac{1}{2}\pi)-\left(\frac{\alpha}{1+\alpha}\right)^{\frac{1}{2}}\mathrm{F}(k,\tfrac{1}{2}\pi)\right],\\ k^2 &= \frac{1-2\alpha}{1-\alpha^2}\quad (0<\alpha\leqslant\tfrac{1}{2}).\end{aligned}\right\} \tag{2·4·53}$$

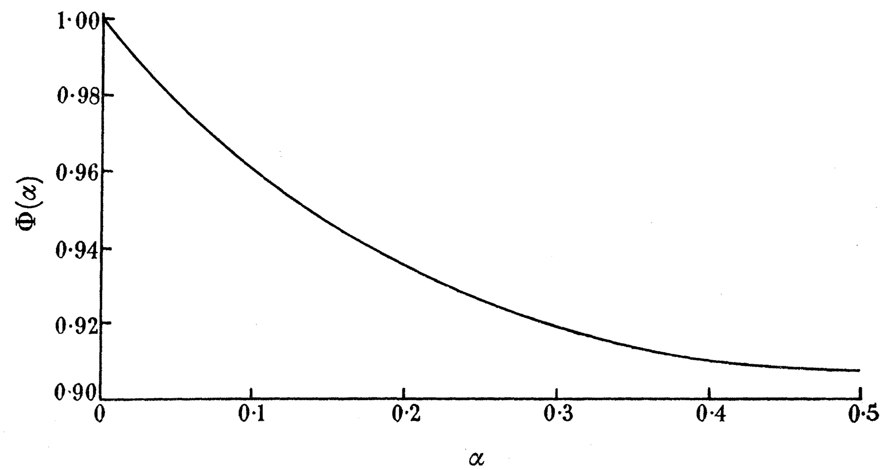


FIGURE 10. Graph of $\Phi(\alpha)$ (defined by (2·4·53)).

The form of $\Phi(\alpha)$ is shown in figure 10. When $\alpha\to 0$, $k^2\to 1$ and $\mathrm{F}(k,\frac{1}{2}\pi)\to\infty$ logarithmically. Hence $\alpha\mathrm{F}(k,\frac{1}{2}\pi)\to 0$ and

$$\lim_{\alpha\to 0}\Phi(\alpha) = 1. \tag{2·4·54}$$

Also when $\alpha = \frac{1}{2}$, $k^2 = 0$ and so

$$\Phi(\tfrac{1}{2}) = (3^{\frac{1}{2}}-3^{-\frac{1}{2}})\,\pi/4 = 0{\cdot}917\,\ldots. \tag{2·4·55}$$

Throughout the whole of its range, Φ departs very little from unity.

A particularly interesting case is when the energy spectrum $\mathrm{E}(u,v)$ is narrow. Let us take axes of (u,v) so that the u axis passes through the centroid $(\bar{u},\bar{v})$, making $\bar{v}=0$. Then on substituting from (1·6·8 and 1·6·9) and retaining only the terms of highest order we find

$$\left.\begin{aligned}\Delta_2 &= \bar{u}^2\mu_{00}\mu_{02},\\ 3H &= \bar{u}^4(\mu_{00}\mu_{04}+3\mu_{02}^2),\\ \Delta_4 &= \bar{u}^6(\mu_{00}\mu_{02}\mu_{04}-\mu_{00}\mu_{03}^2-\mu_{02}^3).\end{aligned}\right\} \tag{2·4·56}$$

Now μ_{00}, μ_{02}, μ_{03}, μ_{04} are moments of the energy spectrum $E_{\frac{1}{2}\pi}$ of a section of the surface at right angles to the mean direction. We have

$$\mu_{02} = (\gamma\bar{u})^2 m_{00},\quad \mu_{03} = b(\gamma\bar{u})^3 m_{00},\quad \mu_{04} = a^2(\gamma\bar{u})^4 m_{00}, \tag{2·4·57}$$

where γ^{-1} is the long-crestedness (defined in § 1·4), a^2 is a non-dimensional parameter (always greater than 1) which represents the *peakedness* of the spectrum $E_{\frac{1}{2}\pi}$, and b is a measure of the asymmetry of $E_{\frac{1}{2}\pi}$ about its mean (but is independent of the angle of skewness β). If we assume $\mu_{03} = 0$ and so $b = 0$, we have

$$\left.\begin{aligned}\Delta_2 &= (\gamma\bar{u}^2)^2\, m_{00}^2,\\ H &= (\gamma\bar{u}^2)^4\, m_{00}^2(1+a^2/3),\\ \Delta_4 &= (\gamma\bar{u}^2)^6\, m_{00}^3(a^2-1)\end{aligned}\right\} \tag{2·4·58}$$

and so from (2·4·32)

$$(l_1, l_2, l_3) = (\gamma\bar{u}^2)^2\, m_{00}(1+a^2/3)^{\frac{1}{2}}\,(\cos\psi_1, \cos\psi_2, \cos\psi_3), \tag{2·4·59}$$

where
$$\cos 3\psi = \frac{a^2-1}{(1+a^2/3)^{\frac{3}{2}}}. \tag{2·4·60}$$

Thus from (2·4·52) we have
$$D_{\text{ma.}} = \gamma\bar{u}^2 C(a), \tag{2·4·61}$$

where
$$C(a) = \frac{1}{2\pi^2}(1+a^2/3)^{\frac{1}{2}}\cos\psi_1\,\Phi\left(-\frac{\cos\psi_2}{\cos\psi_1}\right), \tag{2·4·62}$$

which is a quantity depending only on the peakedness a. For a given peakedness, $D_{\text{ma.}}$ is proportional to the square of the wave-number of the carrier wave and inversely proportional to the long-crestedness γ^{-1}. To illustrate the effect of varying peakedness, $C(a)$ has been computed for a number of different values of a, including some interesting special cases. The results are given in table 1.

TABLE 1

a^2	$C(a)$	a^2	$C(a)$	a^2	$C(a)$
1	0·0507	4	0·0695	8	0·0880
$\frac{9}{5}$	0·0562	5	0·0747	9	0·0919
2	0·0578	6	0·0794	10	0·0956
3	0·0639	7	0·0838	20	0·1265

$a^2 = 1$ corresponds to a pair of intersecting wave trains, for then Δ_4 vanishes (by (2·4·54)), which is the condition for the spectrum to degenerate into two one-dimensional spectra (equation (1·3·7)). In the limit as $a^2 \to 1$ we find from (2·4·60) that

$$(\psi_1, \psi_2, \psi_3) = (\pm\tfrac{1}{6}\pi, \pm\tfrac{1}{2}\pi, \pm\tfrac{5}{6}\pi),$$

and hence

$$C = \frac{1}{2\pi^2} = 0·05066\ldots. \tag{2·4·63}$$

This is what we should expect, for the wavelengths of the pattern in the u direction and the v direction are $2\pi/\bar{u}$ and $2\pi/\gamma\bar{u}$ respectively. Reference to figure 9 *a* will show that each maximum is at the centre of a parallelogram (bounded by troughs) whose diagonals are of length $2\pi/\bar{u}$ and $2\pi/\gamma\bar{u}$, and whose area is therefore $2\pi^2/\gamma\bar{u}^2$. The density of maxima is the reciprocal of this area, i.e. $\gamma\bar{u}^2/2\pi^2$.

The case $a^2 = \frac{9}{5}$ has been included, since this is the peakedness of a low-pass spectrum, when the wave energy is uniformly distributed with regard to direction over a narrow sector. $a^2 = 3$ corresponds to a normal distribution of energy with regard to direction. Another special case is $a^2 = 9$, when $(\psi_1, \psi_2, \psi_3) = (0, \pm\frac{2}{3}\pi, \pm\frac{2}{3}\pi)$, and so

$$C = \frac{1}{4\pi}(3^{\frac{1}{2}} - 3^{-\frac{1}{2}}) = 0·09189\ldots. \tag{2·4·64}$$

Finally as $a \to \infty$ we find

$$C \sim \frac{a}{4\pi^2} \to \infty. \tag{2·4·65}$$

On the whole, however, the variation of C with a is slight. As a^2 increases from 0 to 10, C is less than doubled.

The density of *specular points*, i.e. points on the surface where the gradient takes a given value, not necessarily zero, may be found similarly. For it is only necessary to replace $p(0, 0, \xi_4, \xi_5, \xi_6)$ in equation (2·4·13) by $p(\xi_2, \xi_3, \xi_4, \xi_5, \xi_6)$, and, from equation (2·4·16), this amounts to multiplying by the exponential factor in (2·1·12). So the density $D_{\text{sp.}}$ of specular points with gradient (ξ_2, ξ_3) is given by

$$D_{\text{sp.}} = 4D_{\text{ma.}} \exp\{-(m_{02}\xi_2^2 - 2m_{11}\xi_2\xi_3 + m_{20}\xi_3^2)/2\Delta_2\}, \tag{2·4·66}$$

where $D_{\text{ma.}}$ is the density of maxima.

2·5. *The velocities of zeros along a line*

In this and the following two sections will be considered some statistical properties of the surface which depend on its motion, that is to say properties involving the time t.

Let $\zeta(x', t)$ denote the curve in which the surface is intersected by a fixed vertical plane in direction θ. Consider the movement of a point where the curve crosses a fixed level ζ. If x' and $x' + \mathrm{d}x'$ are its co-ordinates at two successive instants of time t and $t + \mathrm{d}t$, then we have

$$0 = \mathrm{d}\zeta = \frac{\partial \zeta}{\partial x'}\mathrm{d}x' + \frac{\partial \zeta}{\partial t}\mathrm{d}t. \tag{2·5·1}$$

Therefore the velocity of the point is given by

$$c = \frac{\mathrm{d}x'}{\mathrm{d}t} = -\frac{\partial\zeta/\partial t}{\partial\zeta/\partial x'}. \tag{2·5·2}$$

Consider now the statistical distribution of c.

Let the variables ζ, $\partial\zeta/\partial x'$, $\partial\zeta/\partial t$ be denoted by ξ_1, ξ_2, ξ_3 respectively, so that $c = -\xi_3/\xi_2$, and let $p(\xi_1, \xi_2, \xi_3)$ denote the joint distribution of ξ_1, ξ_2, ξ_3 at an arbitrary point x' on the plane section. The probability distribution of ξ_2, ξ_3 at points x' where ξ_1 takes a given value will be denoted by $p(\xi_2, \xi_3)_{\xi_1}$. This may be found as follows. If $(x', x' + \mathrm{d}x')$ is any fixed interval of distance, the probability of ξ_1 taking the given value in $(x', x' + \mathrm{d}x')$ is

$$N_0(\xi_1)\,\mathrm{d}x' \tag{2·5·3}$$

(evaluated in § 2·2). But if, at the point x', the variables ξ_1, ξ_2, ξ_3 lie in certain ranges of width $\mathrm{d}\xi_1$, $\mathrm{d}\xi_2$, $\mathrm{d}\xi_3$, then we have

$$\mathrm{d}\xi_1 = |\xi_2|\,\mathrm{d}x', \tag{2·5·4}$$

so that the probability of ξ_1 taking the given value in $(x', x' + \mathrm{d}x')$ and of ξ_2, ξ_3 lying in the given range is

$$p(\xi_1, \xi_2, \xi_3)\,\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 = p(\xi_1, \xi_2, \xi_3)\,|\xi_1|\,\mathrm{d}x'\mathrm{d}\xi_2\mathrm{d}\xi_3. \tag{2·5·5}$$

The probability of ξ_2, ξ_3 lying in the ranges $\mathrm{d}\xi_2$, $\mathrm{d}\xi_3$ *given that* ξ_1 crosses the given level in $(x', x' + \mathrm{d}x')$ is the quotient of (2·5·5) and (2·5·3). Hence

$$p(\xi_2, \xi_3)_{\xi_1} = \frac{p(\xi_1, \xi_2, \xi_3)\,|\xi_2|}{N_0(\xi_1)}. \tag{2·5·6}$$

Now the matrix of correlations for ξ_1, ξ_2, ξ_3 is

$$(\Xi_{ij}) = \begin{pmatrix} m_0 & 0 & 0 \\ 0 & m_2 & m_1' \\ 0 & m_1' & m_0'' \end{pmatrix}, \tag{2·5·7}$$

and therefore

$$p(\xi_1, \xi_2, \xi_3) = \frac{1}{(2\pi)^{\frac{3}{2}} (m_0 \Delta_{2'})^{\frac{1}{2}}} \exp\{-\xi_1^2/2m_0\} \exp\{-(m_0''\xi_2^2 - 2m_1'\xi_2\xi_3 + m_2\xi_3^2)/2\Delta_{2'}\}, \tag{2·5·8}$$

where

$$\Delta_{2'} = m_2 m_0'' - m_1'^2. \tag{2·5·9}$$

From this and (2·2·12) we have

$$p(\xi_2, \xi_3)_{\xi_1} = \frac{1}{2(2\pi)^{\frac{1}{2}} (m_2 \Delta_{2'})^{\frac{1}{2}}} |\xi_2| \exp\{-(m_0''\xi_2^2 - 2m_1'\xi_2\xi_3 + m_2\xi_3^2)/2\Delta_{2'}\}. \tag{2·5·10}$$

We require now the statistical distribution of $-\xi_3/\xi_2$. Writing

$$-\xi_3/\xi_2 = c, \quad \xi_3 = c' \tag{2·5·11}$$

in (2·5·10), so that

$$\frac{\partial(c, c')}{\partial(\xi_2, \xi_3)} = \frac{\xi_3}{\xi_2^2} = \frac{c^2}{c'}, \tag{2·5·12}$$

we have

$$p(c, c')_{\xi_1} = \frac{1}{2(2\pi)^{\frac{1}{2}} (m_2 \Delta_{2'})^{\frac{1}{2}}} \left|\frac{c'^2}{c^3}\right| \exp\{-(m_0''/c^2 + 2m_1'/c + m_2)\, c'^2/2\Delta_{2'}\}. \tag{2·5·13}$$

The distribution of c is found by integrating with respect to c' from $-\infty$ to ∞. Thus

$$p(c)_{\xi_1} = \frac{1}{2} \frac{\Delta_{2'}/m_2^{\frac{1}{2}}}{(m_0'' + 2m_1'c + m_2c^2)^{\frac{3}{2}}}, \tag{2·5·14}$$

or

$$p(c)_{\xi_1} = \frac{1}{2} \frac{\Delta_{2'}/m_2^2}{[(c-\bar{c})^2 + \Delta_{2'}/m_2^2]^{\frac{3}{2}}}, \tag{2·5·15}$$

where

$$\bar{c} = -m_1'/m_2. \tag{2·5·16}$$

This distribution has a maximum or mode when $c = \bar{c}$ and is symmetrical about this mean value. The second moment of the distribution is divergent, but the interquartile range is given by

$$\frac{2}{\sqrt{3}} \frac{\Delta_{2'}^{\frac{1}{2}}}{m_2}. \tag{2·5·17}$$

It will be seen that the distribution (2·5·15) is independent of the height ξ_1 at which the velocity is measured (provided this height is constant).

We may consider similarly the distribution of velocities of points on the curve having a given gradient (say, zero). The velocity of such a point is given by

$$c_1 = -\frac{\partial^2\zeta/\partial x'\,\partial t}{\partial^2\zeta/\partial x'^2}. \tag{2·5·18}$$

Hence the probability distribution is the same as for the velocities of zeros, except that the index of each of the moments is increased by two. Thus

$$p(c_1) = \frac{1}{2} \frac{\Delta_{4'}/m_4^2}{[(c-\bar{c}_1)^2 + \Delta_{4'}/m_4^2]^{\frac{3}{2}}}, \tag{2·5·19}$$

where

$$\Delta_{4'} = m_4 m_2'' - m_3'^2 \tag{2·5·20}$$

and

$$\bar{c}_1 = -m_3'/m_4. \tag{2·5·21}$$

This is a distribution with mean $\bar{c}_1$ and interquartile range equal to

$$\frac{2}{\sqrt{3}}\frac{\Delta_{4'}^{\frac{1}{2}}}{m_4}. \tag{2·5·22}$$

Similar distributions can be written down for the velocities of points having given higher derivatives of ζ.

Let us interpret the above results for a narrow spectrum. Without loss of generality we may take the u axis to pass through the centroid. On expanding in a Taylor series about this point we have

$$\left.\begin{aligned} m_n &= u'^n\mu_{00}+\frac{1}{2}\left(\mu_{20}\frac{\partial^2}{\partial u^2}+2\mu_{11}\frac{\partial^2}{\partial u\,\partial v}+\mu_{02}\frac{\partial^2}{\partial v^2}\right)u'^n+\dots,\\ m_n' &= u'^n\sigma\mu_{00}+\frac{1}{2}\left(\mu_{20}\frac{\partial^2}{\partial u^2}+2\mu_{11}\frac{\partial^2}{\partial u\,\partial v}+\mu_{02}\frac{\partial^2}{\partial v^2}\right)(u'^n\sigma)+\dots,\\ m_n'' &= u'^n\sigma^2\mu_{00}+\frac{1}{2}\left(\mu_{20}\frac{\partial^2}{\partial u^2}+2\mu_{11}\frac{\partial^2}{\partial u\,\partial v}+\mu_{02}\frac{\partial^2}{\partial v^2}\right)(u'^n\sigma^2)+\dots,\end{aligned}\right\} \tag{2·5·23}$$

where u', v' and σ are to be evaluated at $(\bar{u}, 0)$. Suppose that $\theta = 0$, that is, let the plane of intersection be taken parallel to the principal direction. Then $u' \equiv u$ and, σ being a function of (u^2+v^2) only,

$$\frac{\partial\sigma}{\partial v} = 0, \quad \frac{\partial^2\sigma}{\partial u\,\partial v} = 0. \tag{2·5·24}$$

From (2·5·16) and (2·5·21) we find first

$$\bar{c} = \bar{c}_1 = -\sigma/\mu, \tag{2·5·25}$$

showing that the mean velocities of zeros and of specular points are equal to the phase velocity of the carrier wave. Also from (2·5·9) and (2·5·20)

$$\left.\begin{aligned}\Delta_{2'} &= \mu_{00}\mu_{20}u^2(\sigma/u-\partial\sigma/\partial u)^2,\\ \Delta_{4'} &= \mu_{00}\mu_{20}u^6(\sigma/u-\partial\sigma/\partial u)^2,\end{aligned}\right\} \tag{2·5·26}$$

so that

$$\frac{\Delta_{2'}^{\frac{1}{2}}}{m_2} = \frac{\Delta_{4'}^{\frac{1}{2}}}{m_4} = \left(\frac{\mu_{20}}{\mu_{00}}\right)^{\frac{1}{2}}\frac{\sigma/u-\partial\sigma/\partial u}{u}. \tag{2·5·27}$$

Thus the interquartile ranges of both the velocities of zeros and the velocities of specular points are equal to

$$\frac{2}{\sqrt{3}}\nu(\bar{c}-\Gamma), \tag{2·5·28}$$

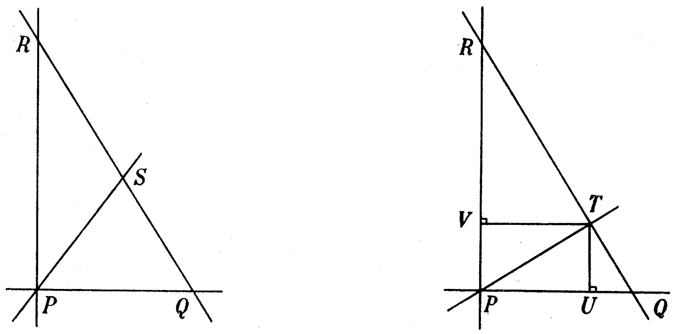
where ν is defined by (1·6·15) and $\Gamma = \partial\sigma/\partial u$ is the group velocity of the carrier wave. Thus we see that the width of the velocity distribution depends both on the r.m.s. width of the spectrum (given by νu) and also on the dispersive properties of the medium. If the medium is non-dispersive, $\Gamma = \bar{c}$ and so (2·5·28) vanishes. This is what we should expect, since in a non-dispersive medium a long-crested disturbance advances without change of form and the zeros and specular points move with uniform velocity in the direction of wave propagation.

For gravity waves in deep water the group velocity is half the wave velocity and so the interquartile range equals $\nu\bar{c}/\sqrt{3}$.

2·6. *The motion of the contours*

Consider first how to define this motion. Let P be a point in the plane lying on the contour $\zeta =$ constant. A moment later the contour at the same level will have moved to a new position, say QSR (see figure 11 *a*). If PQ and PR are axes parallel to Ox and Oy, the rates at which PQ and PR are increasing, which we denote by c_x and c_y, define the local displacement of the contour uniquely, and we have

$$(c_x, c_y) = \left(-\frac{\partial\zeta/\partial t}{\partial\zeta/\partial x}, -\frac{\partial\zeta/\partial t}{\partial\zeta/\partial y}\right). \tag{2·6·1}$$



(*a*) (*b*)

FIGURE 11. Definition of the motion of a contour (*a*) by its intercept on an arbitrary line, (*b*) by its normal displacement.

However, if we take a line through P in an arbitrary direction θ, and if this line intersects the displaced contour in S (figure 11), then it may be shown that

$$\frac{1}{PS} = \frac{1}{PQ}\cos\theta + \frac{1}{PR}\sin\theta, \tag{2·6·2}$$

and so if c is the rate at which the intercept PS is increasing

$$\frac{1}{c} = \frac{1}{c_x}\cos\theta + \frac{1}{c_y}\sin\theta. \tag{2·6·3}$$

This shows that the reciprocal quantity $(1/c_x, 1/c_y)$ is transformed like a vector, but not (c_x, c_y) itself. It is therefore more appropriate, and in fact more convenient, to consider the distribution of

$$(\kappa_x, \kappa_y) = (1/c_x, 1/c_y) = \left(-\frac{\partial\zeta/\partial x}{\partial\zeta/\partial t}, -\frac{\partial\zeta/\partial y}{\partial\zeta/\partial t}\right), \tag{2·6·4}$$

rather than the distribution of (c_x, c_y). However, each may be derived from the other by a simple substitution. For since

$$\frac{\partial(\kappa_x, \kappa_y)}{\partial(c_x, c_y)} = \frac{1}{c_x^2 c_y^2} = \kappa_x^2 \kappa_y^2, \tag{2·6·5}$$

we have

$$p(c_x, c_y) = \frac{1}{c_x^2 c_y^2} p(\kappa_x, \kappa_y). \tag{2·6·6}$$

The distribution of the velocities of the contours normal to themselves can also be found. In figure 11 *b*, T is the foot of the perpendicular from P to QR, and TU, TV are drawn perpendicular to PQ, PR. It can be shown that

$$PU = \frac{PQ\,PR^2}{QR^2}, \quad PV = \frac{PQ^2\,PR}{QR^2}, \tag{2·6·7}$$

and hence the components of the normal velocity are

$$(q_x, q_y) = \left(\frac{c_x c_y^2}{c_x^2+c_y^2}, \frac{c_x^2 c_y}{c_x^2+c_y^2}\right) = \left(\frac{\kappa_x}{\kappa_x^2+\kappa_y^2}, \frac{\kappa_y}{\kappa_x^2+\kappa_y^2}\right). \tag{2·6·8}$$

Solving, we have

$$(\kappa_x, \kappa_y) = \left(\frac{q_x}{q_x^2+q_y^2}, \frac{q_y}{q_x^2+q_y^2}\right), \quad \frac{\partial(\kappa_x, \kappa_y)}{\partial(q_x, q_y)} = -\frac{1}{(q_x^2+q_y^2)^2}, \tag{2·6·9}$$

and so

$$p(q_x, q_y) = \frac{1}{(q_x^2+q_y^2)^2} p(\kappa_x, \kappa_y). \tag{2·6·10}$$

Let us write

$$\zeta, \frac{\partial \zeta}{\partial x}, \frac{\partial \zeta}{\partial y}, \frac{\partial \zeta}{\partial t} = \xi_1, \xi_2, \xi_3, \xi_4, \tag{2·6·11}$$

and let $p(\xi_1, \xi_2, \xi_3, \xi_4)$ denote the joint distribution of ξ_1, ξ_2, ξ_3, ξ_4 at an arbitrary point P in the (x, y) plane. We require the joint distribution of ξ_2, ξ_3, ξ_4 at points distributed uniformly along the contour $\xi_1 =$ constant. Let this be denoted by $p(\xi_2, \xi_3, \xi_4)_{\xi_1}$. To find this distribution let dA be a small area surrounding P. If ξ_2, ξ_3, ξ_4 at P are restricted to lie in certain ranges of width dξ_2, dξ_3, dξ_4, the contribution of the area dA to the distribution over these ranges is, by the argument of § 2·3,

$$p(\xi_1, \xi_2, \xi_3, \xi_4)\, \alpha\, \mathrm{d}A\, \mathrm{d}\xi_2\, \mathrm{d}\xi_3\, \mathrm{d}\xi_4, \tag{2·6·12}$$

where $\alpha = (\xi_2^2+\xi_3^2)^{\frac{1}{2}}$. But the total expectation of contour length over the area dA is $\bar{s}$dA (see § 2·3). Hence we have

$$p(\xi_2, \xi_3, \xi_4)_{\xi_1} = \frac{(\xi_2^2+\xi_3^2)^{\frac{1}{2}} p(\xi_1, \xi_2, \xi_3, \xi_4)}{\bar{s}}. \tag{2·6·13}$$

Now by § 2·1, the elevation ξ_1 is uncorrelated with the first derivatives ξ_2, ξ_3, ξ_4. Therefore

$$p(\xi_1, \xi_2, \xi_3, \xi_4) = p(\xi_1)\, p(\xi_2, \xi_3, \xi_4), \tag{2·6·14}$$

where $p(\xi_1)$ is given by (2·1·8). The matrix of correlations for (ξ_2, ξ_3, ξ_4) is

$$(\Xi_{ij}) = \begin{pmatrix} m_{20} & m_{11} & m'_{10} \\ m_{11} & m_{02} & m'_{01} \\ m'_{10} & m'_{01} & m''_{00} \end{pmatrix}. \tag{2·6·15}$$

Hence

$$p(\xi_2, \xi_3, \xi_4) = \frac{1}{(2\pi)^{\frac{3}{2}} \Delta_3^{\frac{1}{2}}} \exp\{-\tfrac{1}{2} M_{ij} \xi_{i+1} \xi_{j+1}\}, \tag{2·6·16}$$

where (M_{ij}) is the inverse matrix to (Ξ_{ij}) and where $\Delta_3 = |\,\Xi_{ij}\,|$. On substituting these values in (2·6·13) we have

$$p(\xi_2, \xi_3, \xi_4)_{\xi_1} = \frac{1}{4\pi(m_{20}+m_{02})^{\frac{1}{2}} \Delta_3^{\frac{1}{2}}} \frac{(1+\gamma^2)^{\frac{1}{2}}}{\mathrm{E}\{\sqrt{(1-\gamma)^2}\}} (\xi_2^2+\xi_3^2)^{\frac{1}{2}} \exp\{-\tfrac{1}{2} M_{ij} \xi_{i+1} \xi_{j+1}\}. \tag{2·6·17}$$

Writing now

$$\kappa_x = -\xi_2/\xi_4, \quad \kappa_y = -\xi_3/\xi_4, \quad \kappa = \xi_4 \tag{2·6·18}$$

so that

$$\frac{\partial(\kappa_x, \kappa_y, \kappa)}{\partial(\xi_2, \xi_3, \xi_4)} = \frac{1}{\xi_4^2} = \frac{1}{\kappa^2}, \tag{2·6·19}$$

we have

$$p(\kappa_x, \kappa_y, \kappa)_{\xi_1} = \frac{1}{4\pi \Delta_3^{\frac{1}{2}} (m_{20}+m_{02})^{\frac{1}{2}}} \frac{(1+\gamma^2)^{\frac{1}{2}}}{\mathrm{E}\{\sqrt{(1-\gamma^2)}\}} |\,\kappa^3\,|\, (\kappa_x^2+\kappa_y^2)^{\frac{1}{2}}$$
$$\times \exp\{-\tfrac{1}{2}\kappa^2(M_{11}\kappa_x^2 + 2M_{12}\kappa_x\kappa_y + M_{22}\kappa_y^2 - 2M_{13}\kappa_x - 2M_{23}\kappa_y + M_{33})\}. \tag{2·6·20}$$

To obtain the distribution of κ_x, κ_y, we eliminate κ by integrating from $-\infty$ to ∞; thus

$$p(\kappa_x, \kappa_y)_{\xi_1} = \frac{1}{\pi\Delta_3^{\frac{1}{2}}(m_{20}+m_{02})^{\frac{1}{2}}} \frac{(1+\gamma^2)^{\frac{1}{2}}}{\mathrm{E}\{\surd(1-\gamma^2)\}} \frac{(\kappa_x^2+\kappa_y^2)^{\frac{1}{2}}}{R^2}, \tag{2·6·21}$$

where
$$R \equiv M_{11}\kappa_x^2 + 2M_{12}\kappa_x\kappa_y + M_{22}\kappa_y^2 - 2M_{13}\kappa_x - 2M_{23}\kappa_y + M_{33}. \tag{2·6·22}$$

We may also write

$$R = M_{11}(\kappa_x - \bar{\kappa}_x)^2 + 2M_{12}(\kappa_x - \bar{\kappa}_x)(\kappa_y - \bar{\kappa}_y) + M_{22}(\kappa_y - \bar{\kappa}_y)^2 + M, \tag{2·6·23}$$

where

$$\left.\begin{aligned} \bar{\kappa}_x &= \frac{M_{13}M_{22} - M_{12}M_{22}}{M_{11}M_{22} - M_{12}^2} = -\frac{m'_{10}}{m''_{00}}, \\ \bar{\kappa}_y &= \frac{M_{11}M_{23} - M_{12}M_{13}}{M_{11}M_{22} - M_{12}^2} = -\frac{m'_{01}}{m''_{00}}, \\ M &= M_{33} - (M_{11}\bar{\kappa}_x^2 + 2M_{12}\bar{\kappa}_x\bar{\kappa}_y + M_{22}\bar{\kappa}_y^2) = \frac{1}{m''_{00}}. \end{aligned}\right\} \tag{2·6·24}$$

The denominator R is thus a symmetrical expression with a maximum at

$$(\bar{\kappa}_x, \bar{\kappa}_y) = \left(-\frac{m'_{10}}{m''_{00}}, -\frac{m'_{01}}{m''_{00}}\right). \tag{2·6·25}$$

The distribution (2·6·22) itself is not in general symmetrical. However, when the spectrum is narrow, R is appreciable only in the neighbourhood of $(\bar{\kappa}_x, \bar{\kappa}_y)$, giving

$$p(\kappa_x, \kappa_y)_{\xi_1} = \frac{1}{\pi m\Delta_3^{\frac{1}{2}}} \frac{(\bar{\kappa}_x^2+\bar{\kappa}_y^2)^{\frac{1}{2}}}{R^2}, \tag{2·6·26}$$

approximately. The curves of constant probability are then the ellipses $R =$ constant. The major axis of each ellipse makes an angle ϖ with the x axis given by

$$\tan 2\varpi = \frac{2M_{12}}{M_{11} - M_{22}}. \tag{2·6·27}$$

Since $p(\kappa_x, \kappa_y)_{\xi_1}$ is proportional to R^{-2}, it may be shown that the fraction of the distribution lying *outside* the ellipse is proportional to R^{-1}. At the centre, $R = M$. Therefore the ellipse enclosing just half the distribution is

$$R = 2M. \tag{2·6·28}$$

The semi-axes of this ellipse are of length

$$r_1, r_2 = \left[\frac{2M}{(M_{11}+M_{22}) \pm \surd\{(M_{11}-M_{22})^2 + 4M_{12}^2\}}\right]^{\frac{1}{2}}. \tag{2·6·29}$$

To interpret these results, let the u axis be taken so as to pass through the centroid $(\bar{u}, \bar{v})$ of the energy spectrum. The spectrum being narrow, we may expand in a Taylor series about $(\bar{u}, 0)$ thus:

$$\left.\begin{aligned} m''_{00} &= \sigma^2\mu_{00} + \frac{1}{2}\left(\mu_{20}\frac{\partial^2}{\partial u^2} + \mu_{02}\frac{\partial^2}{\partial v^2}\right)\sigma^2 + \dots, \\ m'_{10} &= u\sigma\mu_{00} + \frac{1}{2}\left(\mu_{20}\frac{\partial^2}{\partial u^2} + \mu_{02}\frac{\partial^2}{\partial v^2}\right)(u\sigma) + \dots, \\ m'_{01} &= \mu_{11}\frac{\partial^2}{\partial u\,\partial v}(v\sigma) + \dots, \end{aligned}\right\} \tag{2·6·30}$$

where $u = \bar{u}$. Making use of (1·6·9) we obtain

$$\left.\begin{aligned} M_{11} &= \frac{\mu_{02}}{\mu_{20}\mu_{02}-\mu_{11}^2}\frac{(\sigma/u)^2}{(\sigma/u-\partial\sigma/\partial u)^2}, \\ M_{12} &= \frac{\mu_{11}}{\mu_{20}\mu_{02}-\mu_{11}^2}\frac{\sigma/u}{\sigma/u-\partial\sigma/\partial u}, \\ M_{22} &= \frac{\mu_{20}}{\mu_{20}\mu_{02}-\mu_{11}^2}, \\ \Delta_3 &= \mu_{00}(\mu_{20}\mu_{02}-\mu_{11}^2)\,u^2(\sigma/u-\partial\sigma/\partial u)^2. \end{aligned}\right\} \tag{2·6·31}$$

Hence
$$(\bar{\kappa}_x, \bar{\kappa}_y) = \left(-\frac{u}{\sigma}, 0\right), \tag{2·6·32}$$

and
$$\tan 2\varpi = \frac{2\mu_{11}\,\sigma/u(\sigma/u-\partial\sigma/\partial u)}{\mu_{20}(\sigma/u-\partial\sigma/\partial u)^2-\mu_{02}(\sigma/u)^2}, \tag{2·6·33}$$

showing that the centre of the distribution is the inverse of the phase velocity, and that $\tan 2\varpi$, like $\tan 2\beta$, is proportional to μ_{11} (cf. equation (1·6·14)). When the spectrum is symmetrical the semi-axes of the distribution are given by

$$r_1 = \left(\frac{\mu_{20}}{\mu_{00}}\right)^{\frac{1}{2}}\left|\frac{\sigma/u-\partial\sigma/\partial u}{\sigma^2/u}\right|, \quad r_2 = \left(\frac{\mu_{02}}{\mu_{00}}\right)^{\frac{1}{2}}\frac{1}{\sigma}, \tag{2·6·34}$$

that is,
$$r_1 = |\,\nu\bar{\kappa}(1-\Gamma\bar{\kappa})\,|, \quad r_2 = |\,\gamma\bar{\kappa}\,|, \tag{2·6·35}$$

where $\bar{\kappa}$, $= -u/\sigma$, is the mean reciprocal velocity; Γ, $= -\partial\sigma/\partial u$, is the corresponding group velocity; $\nu = (\mu_{20}/\mu_{00}u^2)^{\frac{1}{2}}$ and $\gamma = (\mu_{02}/\mu_{00}u^2)^{\frac{1}{2}}$. This shows that $r_2/\bar{\kappa}$, which represents the width of the distribution perpendicular to the principal direction of the waves, depends only on the long-crestedness γ^{-1} and is proportional to γ. On the other hand, $r_2/\bar{\kappa}$, which represents the width of the distribution parallel to the principal direction, depends not only on the r.m.s. width of the spectrum, represented by ν, but also on the dispersive properties of the medium. For gravity waves in deep water $\Gamma\bar{\kappa} = \frac{1}{2}$ and so

$$r_1 = \tfrac{1}{2}|\,\nu\bar{\kappa}\,|. \tag{2·6·36}$$

2·7. *The velocities of specular points*

A specular point on the surface is defined as a point where the two components of the gradient take given values. Such points would be indicated to a distant observer as the points where light was reflected from a distant source. In § 2·4 we deduced the mean density of such points per unit area; let us now consider the statistical distribution of their velocities.

If (x, y) are the co-ordinates at time t of a point whose components of gradient

$$\frac{\partial\zeta}{\partial x}, \frac{\partial\zeta}{\partial y} = \xi_2, \xi_3 \tag{2·7·1}$$

are fixed. At a subsequent time $t+\mathrm{d}t$ the point will have moved to a position $(x+\mathrm{d}x, y+\mathrm{d}y)$, where

$$\left.\begin{aligned} 0 &= \mathrm{d}\left(\frac{\partial\zeta}{\partial x}\right) = \frac{\partial^2\zeta}{\partial x^2}\mathrm{d}x + \frac{\partial^2\zeta}{\partial x\,\partial y}\mathrm{d}y + \frac{\partial^2\zeta}{\partial x\,\partial t}\mathrm{d}t, \\ 0 &= \mathrm{d}\left(\frac{\partial\zeta}{\partial y}\right) = \frac{\partial^2\zeta}{\partial x\,\partial y}\mathrm{d}x + \frac{\partial^2\zeta}{\partial y^2}\mathrm{d}y + \frac{\partial^2\zeta}{\partial y\,\partial t}\mathrm{d}t. \end{aligned}\right\} \tag{2·7·2}$$

The ratios

$$\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t} = c_x, c_y \tag{2·7·3}$$

are the required velocities. Writing

$$\left.\begin{aligned} \frac{\partial^2\zeta}{\partial x^2}, \frac{\partial^2\zeta}{\partial x\,\partial y}, \frac{\partial^2\zeta}{\partial y^2} &= \xi_4, \xi_5, \xi_6, \\ \frac{\partial^2\zeta}{\partial x\,\partial t}, \frac{\partial^2\zeta}{\partial y\,\partial t} &= \xi_7, \xi_8 \end{aligned}\right\} \tag{2·7·4}$$

in (2·7·2) we have

$$\left.\begin{aligned} \xi_4 c_x + \xi_4 c_y &= -\xi_7, \\ \xi_5 c_x + \xi_6 c_y &= -\xi_8. \end{aligned}\right\} \tag{2·7·5}$$

Since the velocities c_x, c_y are given in terms of $\xi_4, \ldots, \xi_8$, we require first the joint distribution $p(\xi_4, \ldots, \xi_8)_{\xi_2, \xi_3}$ of these quantities at points where ξ_2, ξ_3 take the given values.

Let $\mathrm{d}A$ denote any small area of the x, y plane, and P a neighbouring point. As usual, $p(\xi_2, \ldots, \xi_8)$ will denote the ordinary distribution of $\xi_2, \ldots, \xi_8$ at P. Now if ξ_2, ξ_3 take the given values at some point in $\mathrm{d}A$, and $\xi_4, \ldots, \xi_8$ are fixed, then (ξ_2, ξ_3) at P lies within a certain region $\mathrm{d}\Sigma$ of area

$$\mathrm{d}\Sigma = |\,\xi_4\xi_6 - \xi_5^2\,|\,\mathrm{d}A \tag{2·7·6}$$

(cf. § 2·4). Hence the probability that ξ_2, ξ_3 take the given values in $\mathrm{d}A$ and that $\xi_4, \ldots, \xi_8$ lie in ranges of width $\mathrm{d}\xi_4, \ldots, \mathrm{d}\xi_8$ respectively is

$$p(\xi_2, \xi_3, \ldots, \xi_8)\,|\,\xi_4\xi_6 - \xi_5^2\,|\,\mathrm{d}A\,\mathrm{d}\xi_4\,\mathrm{d}\xi_5 \ldots \mathrm{d}\xi_8. \tag{2·7·7}$$

But the total probability of ξ_2, ξ_3 taking the given values in $\mathrm{d}A$ is

$$D_{\text{sp.}}\,\mathrm{d}A, \tag{2·7·8}$$

where $D_{\text{sp.}}$ is the density of specular points with gradient (ξ_2, ξ_3). Therefore the probability that $\xi_4 \ldots \xi_8$ lie in their respective ranges *given* that there is a specular point in $\mathrm{d}A$ is the quotient of (2·7·7) and (2·7·8), that is

$$\frac{p(\xi_2, \ldots, \xi_8)\,|\,\xi_4\xi_6 - \xi_5^2\,|\,\mathrm{d}\xi_4 \ldots \mathrm{d}\xi_8}{D_{\text{sp.}}}. \tag{2·7·9}$$

In other words

$$p(\xi_4, \ldots, \xi_8)_{\xi_2, \xi_3} = \frac{p(\xi_2, \ldots, \xi_8)\,|\,\xi_4\xi_6 - \xi_5^2\,|}{D_{\text{sp.}}}. \tag{2·7·10}$$

Now the first derivatives ξ_2, ξ_3 are statistically independent of the second derivatives $\xi_4 \ldots \xi_8$. Therefore

$$p(\xi_2, \ldots, \xi_8) = p(\xi_2, \xi_3)\,p(\xi_4, \ldots, \xi_8), \tag{2·7·11}$$

where $p(\xi_2, \xi_3)$ is the ordinary distribution of ξ_2, ξ_3 given by (2·1·12) and $p(\xi_4, \ldots, \xi_8)$ is the ordinary distribution of $\xi_4, \ldots, \xi_8$. The matrix of correlations for $\xi_4, \ldots, \xi_8$ is

$$(\Xi_{ij}) = \left(\begin{array}{ccc:cc} m_{40} & m_{31} & m_{22} & m'_{30} & m'_{21} \\ m_{31} & m_{22} & m_{13} & m'_{21} & m'_{12} \\ m_{22} & m_{13} & m_{04} & m'_{12} & m'_{03} \\ \hdashline m'_{30} & m'_{21} & m'_{12} & m'_{20} & m''_{11} \\ m'_{21} & m'_{12} & m'_{03} & m''_{11} & m''_{02} \end{array}\right), \tag{2·7·12}$$

and so
$$p(\xi_4, \ldots, \xi_8) = \frac{1}{(2\pi)^{\frac{5}{2}} \Delta_5^{\frac{1}{2}}} \exp\{-\tfrac{1}{2}M_{ij}\xi_{i+3}\xi_{j+3}\}, \tag{2·7·13}$$
where (M_{ij}) is the inverse matrix to (Ξ_{ij}) and where
$$\Delta_5 = |\Xi_{ij}|. \tag{2·7·14}$$
Substituting in (2·7·10) we have then
$$p(\xi_4, \ldots, \xi_8)_{\xi_2, \xi_3} = \frac{1}{4(2\pi)^{\frac{7}{2}} (\Delta_2\Delta_5)^{\frac{1}{2}} D_{\text{ma.}}} |\xi_4\xi_6 - \xi_5^2| \exp\{-\tfrac{1}{2}M_{ij}\xi_{i+3}\xi_{j+3}\}, \tag{2·7·15}$$
where $D_{\text{ma.}}$ is given by (2·4·51).

The distribution of the velocities c_x, c_y is now obtained from the relations (2·7·5). If we write also
$$c_1, c_2, c_3 = \xi_4, \xi_5, \xi_6 \tag{2·7·16}$$
and transform to the variables c_x, c_y, c_1, c_2, c_3 we have
$$\frac{\partial(\xi_4, \ldots, \xi_8)}{\partial(c_x, c_y, c_1, c_2, c_3)} = \xi_4\xi_6 - \xi_5^2 = c_1c_3 - c_2^2. \tag{2·7·17}$$
Hence
$$p(c_x, c_y, c_1, c_2, c_3)_{\xi_2, \xi_3} = \frac{1}{4(2\pi)^{\frac{7}{2}} (\Delta_2\Delta_5)^{\frac{1}{2}} D_{\text{ma.}}} (c_1c_3 - c_2^2)^2 \exp\{-\tfrac{1}{2}N_{ij}c_ic_j\}, \tag{2·7·18}$$
where (N_{ij}) is the (3×3) matrix whose elements are
$$\left.\begin{aligned}
N_{11} &= M_{44}c_x^2 - 2M_{41}c_x + M_{11},\\
N_{22} &= M_{55}c_x^2 + 2M_{45}c_xc_y + M_{44}c_y^2 - 2M_{52}c_x - 2M_{42}c_y + M_{22},\\
N_{33} &= M_{55}c_y^2 - 2M_{53}c_y + M_{33},\\
N_{23} &= M_{55}c_xc_y + M_{45}c_y^2 - M_{53}c_x - (M_{43}+M_{52})\,c_y + M_{23},\\
N_{31} &= M_{45}c_xc_y - M_{43}c_x - M_{51}c_y + M_{31},\\
N_{12} &= M_{45}c_x^2 + M_{44}c_xc_y - (M_{42}+M_{51})\,c_x - M_{41}c_y + M_{12}.
\end{aligned}\right\} \tag{2·7·19}$$
On eliminating c_1, c_2, c_3 by integration between $\pm\infty$ we have
$$p(c_x, c_y)_{\xi_2, \xi_3} = \frac{1}{4(2\pi)^{\frac{7}{2}} (\Delta_2\Delta_5)^{\frac{1}{2}} D_{\text{ma.}}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (c_1c_3 - c_2^2)^2 \exp\{-\tfrac{1}{2}N_{ij}c_ic_j\}\,dc_1\,dc_2\,dc_3. \tag{2·7·20}$$
The matrix (N_{ij}) is positive-definite. For, if any real values of c_1, c_2, c_3, not all zero existed which made the quadratic form $N_{ij}c_ic_j$ zero or negative, a corresponding set of values of $\xi_4 \ldots \xi_8$ could be found from (2·7·5) which made $M_{ij}\xi_{i+3}\xi_{j+3}$ zero or negative. But this is impossible, since (M_{ij}) is positive-definite. Therefore (N_{ij}) is positive-definite. Therefore by a real linear transformation of variables $c_1, c_2, c_3 \to \eta_1, \eta_2, \eta_3$ we have
$$\left.\begin{aligned}
N_{ij}c_ic_j &= \eta_1^2 + \eta_2^2 + \eta_3^2,\\
c_1c_3 - c_2^2 &= \mathbf{l}_1\eta_1^2 + \mathbf{l}_2\eta_2^2 + \mathbf{l}_3\eta_3^2,
\end{aligned}\right\} \tag{2·7·21}$$
where $\mathbf{l}_1$, $\mathbf{l}_2$, $\mathbf{l}_3$ are roots of the equation
$$|\sigma_{ij} - \mathbf{l}N_{ij}| = 0. \tag{2·7·22}$$
As in § 2·4 we have then
$$p(c_x, c_y)_{\xi_2, \xi_3} = \frac{1}{4(2\pi)^{\frac{7}{2}} (\Delta_2\Delta_5 |N_{ij}|)^{\frac{1}{2}} D_{\text{ma.}}} \mathbf{I}(\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3), \tag{2·7·23}$$

where

$$\mathbf{I}(\mathbf{l}_1,\mathbf{l}_2,\mathbf{l}_3)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\mathbf{l}_1\eta_1^2+\mathbf{l}_2\eta_2^2+\mathbf{l}_3\eta_3^2)^2\exp\{-\tfrac{1}{2}(\eta_1^2+\eta_2^2+\eta_3^2)\}\,\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,\mathrm{d}\eta_3. \quad (2\cdot7\cdot24)$$

This integral is much easier to evaluate than the similar integral $I(l_1, l_2, l_3)$ of equation (2·4·36) on account of the factor in the integrand being squared. In fact we have immediately

$$\begin{aligned}\mathbf{I}(\mathbf{l}_1,\mathbf{l}_2,\mathbf{l}_3)&=(2\pi)^{\frac{3}{2}}[3(\mathbf{l}_1^2+\mathbf{l}_2^2+\mathbf{l}_3^2)+2(\mathbf{l}_2\mathbf{l}_3+\mathbf{l}_3\mathbf{l}+\mathbf{l}_1\mathbf{l}_2)].\\&=(2\pi)^{\frac{3}{2}}[3(\mathbf{l}_1+\mathbf{l}_2+\mathbf{l}_3)^2-4(\mathbf{l}_2\mathbf{l}_3+\mathbf{l}_3\mathbf{l}_1+\mathbf{l}_1\mathbf{l}_2)].\end{aligned} \quad (2\cdot7\cdot25)$$

Therefore $$p(c_x,c_y)_{\xi_2,\xi_3}=\frac{1}{16\pi^2(\Delta_2\Delta_5\,|\,N_{ij}\,|)^{\frac{1}{2}}D_{\text{ma.}}}[3(\sum_i\mathbf{l}_i)^2-4\sum_{i\neq j}\mathbf{l}_i\mathbf{l}_j]. \quad (2\cdot7\cdot26)$$

Equation (2·7·22) on expansion becomes

$$N\mathbf{l}^3-(\tfrac{1}{2}n_{13}-n_{22}+\tfrac{1}{2}n_{31})\,\mathbf{l}^2+(\tfrac{1}{2}N_{13}-\tfrac{1}{4}N_{22}+\tfrac{1}{2}N_{31})\,\mathbf{l}-\tfrac{1}{4}=0, \quad (2\cdot7\cdot27)$$

where $$N=|N_{ij}| \quad (2\cdot7\cdot28)$$

and $$n_{13}=N_{21}N_{32}-N_{22}N_{31}=n_{31}, \qquad n_{22}=N_{11}N_{33}-N_{22}^2. \quad (2\cdot7\cdot29)$$

Therefore $$\sum_i\mathbf{l}_i=\frac{n_{13}-n_{22}}{N}, \qquad \sum_{i\neq j}\mathbf{l}_i\mathbf{l}_j=\frac{N_{13}-\frac{1}{4}N_{22}}{N}, \quad (2\cdot7\cdot30)$$

and so finally

$$p(c_x,c_y)_{\xi_1,\xi_2}=\frac{1}{16\pi^2(\Delta_2\Delta_5)^{\frac{1}{2}}D_{\text{ma.}}}\,\frac{3(n_{13}-n_{22})^2+(N_{22}-4N_{13})\,N}{N^{\frac{5}{2}}}. \quad (2\cdot7\cdot31)$$

It will be seen that in general the quantities N_{ij}, n_{ij} and N which occur in this expression are polynomials in c_x, c_y of degree 2, 4 and 6 respectively.

As before, we may study this distribution in the special case when the energy spectrum is narrow and has symmetry about the principal direction. Taking the u axis along the line of symmetry we have $m_{pq}=0$ whenever q is odd, and so

$$(\Xi_{ij})=\left(\begin{array}{ccc|cc}m_{40}&0&m_{22}&m'_{30}&0\\0&m_{22}&0&0&m'_{12}\\m_{22}&0&m_{04}&m'_{12}&0\\\hline m'_{30}&0&m'_{12}&m''_{20}&0\\0&m'_{12}&0&0&m''_{02}\end{array}\right). \quad (2\cdot7\cdot32)$$

The reciprocal matrix (M_{ij}) is given by

$$(M_{ij})=\begin{pmatrix}A_{11}&0&A_{12}&A_{13}&0\\0&B_{11}&0&0&B_{12}\\A_{21}&0&A_{22}&A_{23}&0\\A_{31}&0&A_{32}&A_{33}&0\\0&B_{21}&0&0&B_{22}\end{pmatrix}, \quad (2\cdot7\cdot33)$$

where $$(A_{ij})=\begin{pmatrix}m_{40}&m_{22}&m'_{30}\\m_{22}&m_{04}&m'_{12}\\m'_{30}&m'_{12}&m''_{20}\end{pmatrix}^{-1}, \qquad (B_{ij})=\begin{pmatrix}m_{22}&m'_{12}\\m'_{12}&m''_{02}\end{pmatrix}^{-1}. \quad (2\cdot7\cdot34)$$

Since the spectrum is narrow, each coefficient may be expanded in a Taylor series about the centroid $(\bar{u}, 0)$. Thus

$$\left.\begin{aligned}
m_{40} &= \mu_{00}u^4 + \tfrac{1}{2}\mu_{20}\frac{\partial^2}{\partial u^2}(u^4) + \ldots,\\
m_{22} &= \mu_{02}u^2 + \mu_{12}\frac{\partial}{\partial u}(u^2) + \tfrac{1}{2}\mu_{22}\frac{\partial^2}{\partial u^2}(u^2),\\
m_{04} &= \mu_{04},\\
m'_{30} &= \mu_{00}u^3\sigma + \frac{1}{2}\left(\mu_{20}\frac{\partial^2}{\partial u^2} + \mu_{02}\frac{\partial^2}{\partial v^2}\right)(u^3\sigma) + \ldots,\\
m'_{12} &= \mu_{02}u\sigma + \mu_{12}\frac{\partial}{\partial u}(u\sigma) + \frac{1}{2}\left(\mu_{22}\frac{\partial^2}{\partial u^2} + \mu_{04}\frac{\partial^2}{\partial v^2}\right)(u\sigma) + \ldots,\\
m''_{20} &= \mu_{00}u^2\sigma^2 + \frac{1}{2}\left(\mu_{20}\frac{\partial^2}{\partial u^2} + \mu_{02}\frac{\partial^2}{\partial v^2}\right)(u^2\sigma^2) + \ldots,\\
m''_{02} &= \mu_{02}\sigma^2 + \mu_{12}\frac{\partial}{\partial u}(\sigma^2) + \frac{1}{2}\left(\mu_{22}\frac{\partial^2}{\partial u^2} + \mu_{04}\frac{\partial^2}{\partial v^2}\right)(\sigma^2) + \ldots,
\end{aligned}\right\} \tag{2·7·35}$$

where $u = \bar{u}$ and $\sigma = \sigma(\bar{u}, 0)$. Thus we find for the determinants of $(A_{ij})^{-1}$ and $(B_{ij})^{-1}$

$$\left.\begin{aligned}
|A_{ij}|^{-1} &= \delta_3 u^6(\sigma/u - \partial\sigma/\partial u)^2,\\
|B_{ij}|^{-1} &= \delta_2 u^2(\sigma/u - \partial\sigma/\partial u)^2,
\end{aligned}\right\} \tag{2·7·36}$$

where
$$\delta_3 = \begin{vmatrix} \mu_{00} & 0 & \mu_{02} \\ 0 & \mu_{20} & \mu_{12} \\ \mu_{02} & \mu_{12} & \mu_{04} \end{vmatrix}, \qquad \delta_2 = \begin{vmatrix} \mu_{02} & \mu_{12} \\ \mu_{12} & \mu_{22} \end{vmatrix}. \tag{2·7·37}$$

(Each zero term in δ_3 could be replaced by μ_{10}.) Further, on evaluating (A_{ij}) and (B_{ij}) and substituting in (2·7·19) we find eventually

$$(N_{ij}) = \frac{1}{u^4}\begin{pmatrix}
\alpha_{22}u^2q_1^2 + 2\alpha_{12}uq_1 + \alpha_{11} & \alpha_{22}u^2q_1q_2 + \alpha_{12}uq_2 & \alpha_{23}u^3q_1 + \alpha_{13}u^2 \\
\alpha_{22}u^2q_1q_2 + \alpha_{12}uq_2 & \alpha_{22}u^2q_2^2 + \beta_{22}u^4q_1^2 + 2\beta_{12}u^3q_1 + \beta_{11}u^2 & \alpha_{23}u^3q_2 + \beta_{22}u^4q_1q_2 + \beta_{12}u^3q_2 \\
\alpha_{23}u^3q_1 + \alpha_{13}u^2 & \alpha_{23}u^3q_2 + \beta_{22}u^4q_1q_2 + \beta_{12}u^3q_2 & \alpha_{33}u^4 + \beta_{22}u^4q_2^2
\end{pmatrix} \tag{2·7·38}$$

where α_{ij} and β_{ij} are the (i,j)th elements of the reciprocal matrices of δ_3 and δ_2 respectively, and

$$q_1,\ q_2 = \frac{c_x + \sigma/u}{\sigma/u - \partial\sigma/\partial u},\ \frac{c_y}{\sigma/u - \partial\sigma/\partial u}. \tag{2·7·39}$$

Thus q_1, q_2 are non-dimensional quantities proportional to the departures of c_x, c_y from their mean values $(-\sigma/u, 0)$. In deriving (2·7·38) it has been assumed that q_1 is of the same order of magnitude as γ, $= (\mu_{02}/u^2\mu_{00})^{\frac{1}{2}}$, but that q_2 is of order 1; this makes the matrix (N_{ij}) more homogeneous. Before proceeding further we may make the additional restrictions

$$\mu_{12} = 0, \quad \mu_{22} = \mu_{20}\mu_{02}/\mu_{00}, \tag{2·7·40}$$

and we may write
$$\mu_{20} = \nu^2u^2\mu_{00}, \quad \mu_{02} = \gamma^2u^2\mu_{00}, \quad \mu_{04} = a^2\gamma^4u^4\mu_{00}, \tag{2·7·41}$$

where ν, γ and a have the same meanings as before, namely, ν^{-1} is a measure of the average length of a group of waves, γ^{-1} is the long-crestedness and a is the peakedness of the spectrum in the v direction. Then (2·7·38) reduces to

$$(N_{ij}) = \frac{1}{u^4\mu_{00}}\begin{pmatrix} (\xi^2+d^2+1) & \xi\eta/\gamma & -d^2/\gamma^2 \\ \xi\eta/\gamma & (\xi^2+\eta^2+1)/\gamma^2 & \xi\eta/\gamma^3 \\ -d^2/\gamma^2 & \xi\eta/\gamma^3 & (\xi^2+\eta^2)/\gamma^4 \end{pmatrix}, \tag{2·7·42}$$

where $$\xi = q_1/\nu, \quad \eta = \gamma q_2/\nu, \quad d^2 = 1/(a^2-1). \tag{2·7·43}$$

Clearly the (i,j)th term of (N_{ij}) is of order $1/\gamma^{i+j}$. Therefore

$$\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3 = \gamma^2 u^4 \mu_{00}\,(\mathbf{l}'_1, \mathbf{l}'_2, \mathbf{l}'_3) \tag{2·7·44}$$

where $\mathbf{l}'_1$, $\mathbf{l}'_2$, $\mathbf{l}'_3$ are the corresponding roots of the equation

$$|\,\sigma_{ij} - \mathbf{l}' N'_{ij}\,| = 0 \tag{2·7·45}$$

and (N'_{ij}) is the non-dimensional matrix

$$(N'_{ij}) = \begin{pmatrix} \xi^2+d^2+1 & \xi\eta & -d^2 \\ \xi\eta & \xi^2+\eta^2+1 & \xi\eta \\ -d^2 & \xi\eta & \eta^2+d^2 \end{pmatrix}. \tag{2·7·46}$$

Also $$|\,N_{ij}\,| = |\,N'_{ij}\,|/(\gamma^6 u^{12}\mu_{00}^3). \tag{2·7·47}$$

So from equation (2·7·26)

$$p(c_x, c_y)_{\xi_2,\xi_3} = \frac{\gamma^7 u^{14}\mu_{00}^{\frac{7}{2}}}{16\pi^2(\Delta_2\Delta_5)^{\frac{1}{2}} D_{\text{ma.}}}\,\frac{3(\sum_i \mathbf{l}'_i)^2 - 4\sum_{i\neq j}\mathbf{l}'_i\mathbf{l}'_j}{|\,N'_{ij}\,|^{\frac{1}{2}}}. \tag{2·7·48}$$

It is convenient to state the solution in terms of the non-dimensional variables

$$\xi, \eta = \frac{c_x + \sigma/u}{\nu(\sigma/u - \partial\sigma/\partial u)}, \frac{\gamma c_y}{\nu(\sigma/u - \partial\sigma/\partial u)}, \tag{2·7·49}$$

so that $$p(\xi,\eta)_{\xi_2,\xi_3} = \frac{\nu^2(\sigma/u - \partial\sigma/\partial u)^2}{\gamma}\,p(c_x, c_y)_{\xi_2,\xi_3}. \tag{2·7·50}$$

First, on expanding the left-hand side of (2·7·45) we have

$$N'\mathbf{l}'^3 + \eta^2\mathbf{l}'^2 - [\tfrac{1}{4}(\xi^2+\eta^2+1) + d^2]\,\mathbf{l}' - \tfrac{1}{4} = 0, \tag{2·7·51}$$

where $$N' = |\,N'_{ij}\,| = (\xi^2+\eta^2+1)\,\eta^2 + \{(\xi+\eta)^2+1\}\{(\xi-\eta)^2+1\}\,d^2, \tag{2·7·52}$$

giving $$\sum_i \mathbf{l}'_i = -\eta^2/N', \qquad \sum_{i\neq j}\mathbf{l}'_i\mathbf{l}'_j = -[\tfrac{1}{4}(\xi^2+\eta^2+1) + d^2]/N'. \tag{2·7·53}$$

Secondly, from (2·4·58), (2·7·36) and (2·7·37),

$$\Delta_2 = \gamma^2 u^4 \mu_{00}^2, \qquad \Delta_5 = |\,A_{ij}\,|^{-1}\,|\,B_{ij}\,|^{-1} = \delta_2\delta_3 u^8(\sigma/u - \partial\sigma/\partial u)^4, \tag{2·7·54}$$

and $$\delta_2\delta_3 = \mu_{20}\mu_{22}\mu_{02}(\mu_{00}\mu_{04} - \mu_{02}^2) = \nu^4\gamma^8 u^{12}\mu_{00}^5(a^2-1). \tag{2·7·55}$$

Thirdly from (2·4·61) $$D_{\text{ma.}} = \gamma u^2 C(a). \tag{2·7·56}$$

(a) $a^2 = 1$

(b) $a^2 = 3$

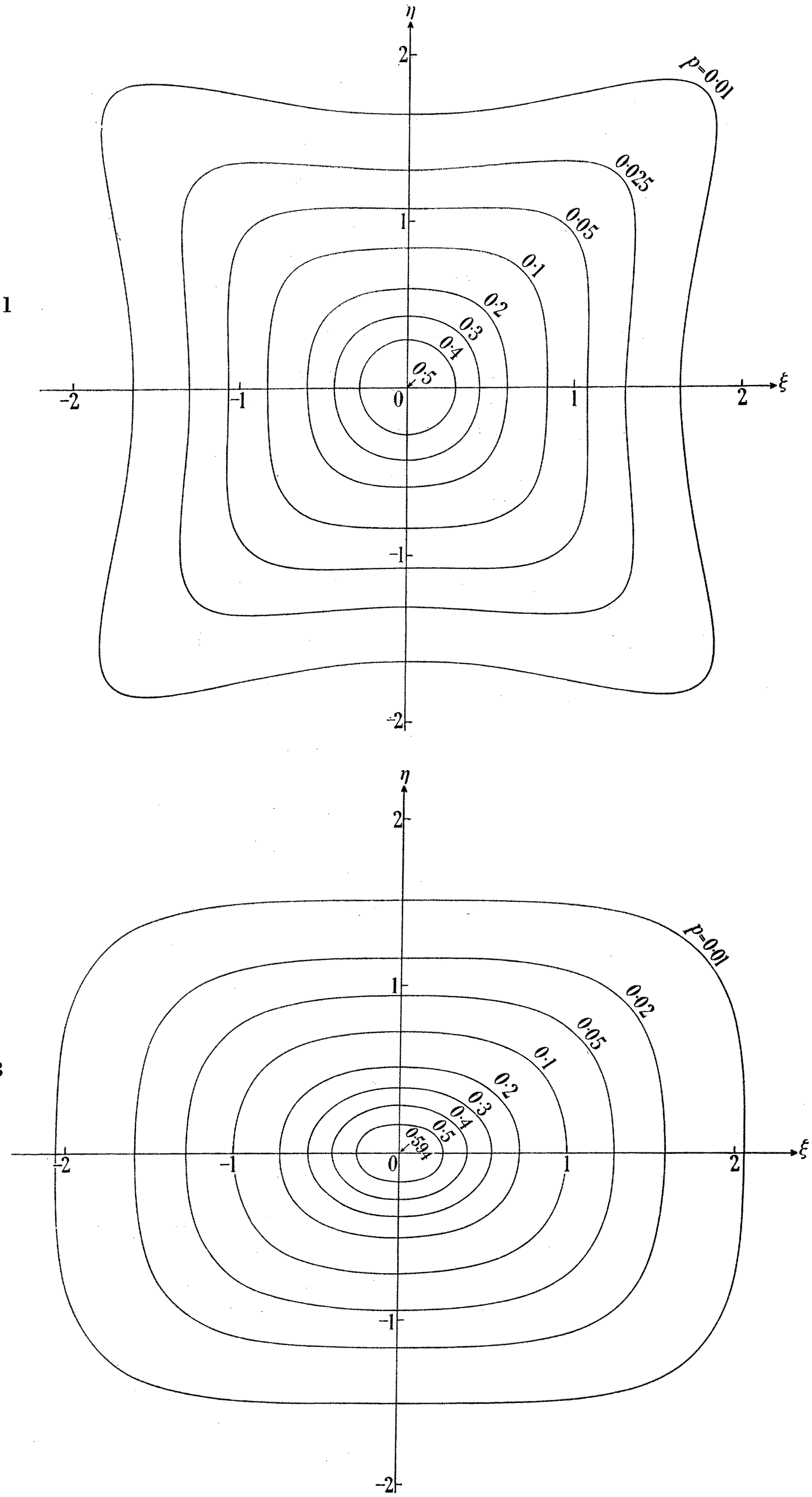


FIGURE 12

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(c) $a^2=\infty$

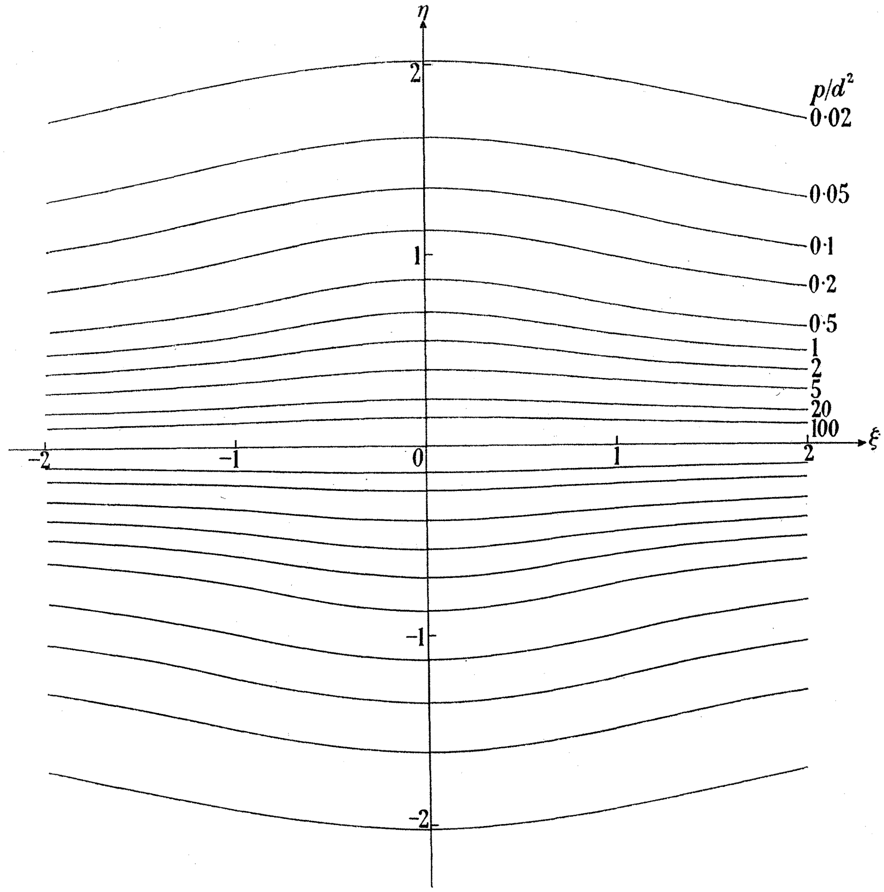

FIGURE 12. The probability distribution of the velocities of specular points, for a narrow spectrum

Therefore, altogether we have

$$p(\xi,\eta)_{\xi_2,\xi_3}=\frac{d}{16\pi^2C(a)}\,\frac{3\eta^4+[(\xi^2+\eta^2+1)+4d^2]\,N'}{N'^{\frac{5}{2}}},\tag{2·7·57}$$

where $C(a)$ is given by (2·4·62) and N' by (2·7·52).

Two special cases are of interest. Suppose first that the surface consists of two systems of long-crested waves, intersecting at a small angle 2γ. As we saw in § 2·4, this corresponds to the limiting case when $a\to 1$ and $d\to\infty$. Equation (2·7·57) then becomes

$$p(\xi,\eta)_{\xi_2,\xi_3}=\frac{1}{2}\,\frac{1}{\{(\xi+\eta)^2+1\}^{\frac{3}{2}}\{(\xi-\eta)^2+1\}^{\frac{3}{2}}}.\tag{2·7·58}$$

The distribution is shown in figure 12*a*. There are two ridges of high probability, in the directions $\xi=\pm\eta$, that is

$$c_x+\sigma/u=\pm\gamma c_y,\tag{2·7·59}$$

or when the vector difference between the specular velocity (c_x, c_y) and the mean velocity $(-\sigma/u, 0)$ is in the direction of the crests of one of the two wave systems.

This particular case may also be derived quite simply as follows. We have seen that the velocities of specular points have the same probability distribution as the velocities of the maxima only. Now with two intersecting systems of long-crested waves, the maxima occur at the points of intersection of the crests of the two systems, and at no other points (see figure 9 *a*). If the point of intersection of two crests has components of velocity c_x, c_y parallel and perpendicular to the mean direction, then the rates of advance of the crests in the two systems of waves are

$$c_1^{(1)} = c_x \cos\gamma + c_y \sin\gamma, \quad c_1^{(2)} = c_x \cos\gamma - c_y \sin\gamma, \tag{2·7·60}$$

where 2γ is the angle between the two wave systems. But the distribution of c_1 for a long-crested system of waves was found in § 2·5 to be

$$p(c_1) = \frac{1}{2}\frac{\nu^2(\sigma/u - \partial\sigma/\partial u)^2}{[(c_1+\sigma/u)^2 + \nu^2(\sigma/u-\partial\sigma/\partial u)^2]^{\frac{3}{2}}}, \tag{2·7·61}$$

where u is measured in the direction of propagation. Since the two systems are independent,

$$p(c_1^{(1)}, c_1^{(2)}) = p(c_1^{(1)})\, p(c_1^{(2)}) \tag{2·7·62}$$

When the angle of separation 2γ is small, ν and u are effectively the same for the two systems and for the combined system. Thus

$$\left.\begin{aligned} c_1^{(1)} + \sigma/u &= (c_x + \sigma/u) + \gamma c_y = (\xi+\eta)\,\nu(\sigma/u - \partial\sigma/\partial u), \\ c_1^{(2)} + \sigma/u &= (c_x + \sigma/u) - \gamma c_y = (\xi-\eta)\,\nu(\sigma/u - \partial\sigma/\partial u). \end{aligned}\right\} \tag{2·7·63}$$

Further,

$$p(\xi,\eta)_{\xi_2,\xi_3} = \left|\frac{\partial(c_1^{(1)}, c_1^{(2)})}{\partial(\xi,\eta)}\right| p(c_1^{(1)}, c_1^{(2)})_{\xi_2,\xi_3}, \tag{2·7·64}$$

and so

$$p(\xi,\eta)_{\xi_2,\xi_3} = 2\nu^2(\sigma/u - \partial\sigma/\partial u)^2\, p(c_1^{(1)})\, p(c_1^{(2)}), \tag{2·7·65}$$

from which (2·7·58) follows.

A second case of interest is that of infinite peakedness: $a \to \infty$ and $d \to 0$. For large values of a, (2·7·57) becomes

$$p(\xi,\eta)_{\xi_2,\xi_3} = \frac{1}{4a^2}\frac{3\eta^2 + (\xi^2+\eta^2+1)^2}{\eta^3(\xi^2+\eta^2+1)^{\frac{5}{2}}}. \tag{2·7·66}$$

This distribution is shown in figure 12 *c*. There is only one ridge of high probability, namely, that in the principal direction of the waves. The expression is valid only asymptotically, as is shown by the presence of the factor $1/4a^2$ and the fact that $\iint p(\xi,\eta)_{\xi_2,\xi_3}\, d\xi\, d\eta$ diverges.

An intermediate case, $a^2 = 3$, $d^2 = \frac{1}{2}$, is shown in figure 12 *b*. This corresponds to a distribution of energy distributed normally with regard to direction, over a narrow range.

2·8. *Properties of the envelope: the number of waves in a group*

We shall now consider briefly some statistical properties of the envelope of the wave surface, as defined in § 1·5. The envelope function ρ is essentially different from the surface elevation ζ, in that ρ is always positive whereas ζ has a mean value zero. Nevertheless, many of the properties of ρ will be seen to be analogous to corresponding properties of ζ.

It is convenient to introduce the auxiliary variables

$$\left.\begin{aligned} \xi_1 &= \rho\cos\phi = \sum_n c_n \cos\{(u_n - \bar{u})\,x + (v_n - \bar{v})\,y + (\sigma_n - \bar{\sigma})\,t + \epsilon_n\}, \\ \xi_2 &= \rho\sin\phi = \sum_n c_n \sin\{(u_n - \bar{u})\,x + (v_n - \bar{v})\,y + (\sigma_n - \bar{\sigma})\,t + \epsilon_n\}, \end{aligned}\right\} \tag{2·8·1}$$

which are the real and imaginary parts of the complex envelope function $\rho\, e^{i\phi}$ (see equation (1·5·6)). ξ_1, ξ_2 have the same form as ζ, each being the sum of an infinite number of sinusoidal

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components with random phase. In fact, the energy spectrum of ξ_1, ξ_2 is the same as that of ζ, but with the origin moved to the mean wave-number $(\bar{u}, \bar{v})$. Thus ξ_1, ξ_2 are normally distributed with mean value zero. Since

$$\overline{\xi_1^2} = \overline{\xi_2^2} = m_{00}, \quad \overline{\xi_1 \xi_2} = 0, \tag{2·8·2}$$

we have

$$p(\xi_1, \xi_2) = \frac{1}{2\pi m_{00}} \exp\{-(\xi_1^2 + \xi_2^2)/2m_{00}\}. \tag{2·8·3}$$

We now transform back to the variables ρ, ϕ. From (2·8·1)

$$\xi_1^2 + \xi_2^2 = \rho^2, \quad \frac{\partial(\xi_1, \xi_2)}{\partial(\rho, \phi)} = \rho, \tag{2·8·4}$$

and so

$$p(\rho, \phi) = \frac{1}{2\pi m_{00}} \rho \exp\{-\rho^2/2m_{00}\}. \tag{2·8·5}$$

This is independent of the phase angle ϕ. The distribution of ρ alone is found by integrating with respect to ϕ from 0 to 2π:

$$p(\rho) = \frac{1}{m_{00}} \rho \exp\{-\rho^2/2m_{00}\}, \tag{2·8·6}$$

which is the well-known Rayleigh distribution.

The joint distribution of ρ, ϕ and their first-order derivatives with respect to x, y may be found as follows. Let

$$\xi_3, \xi_4 = \frac{\partial \xi_1}{\partial x}, \frac{\partial \xi_1}{\partial y}; \quad \xi_5, \xi_6 = \frac{\partial \xi_2}{\partial x}, \frac{\partial \xi_2}{\partial y}. \tag{2·8·7}$$

The matrix of correlations for $\xi_1, \ldots, \xi_6$ is

$$(\Xi_{ij}) = \left(\begin{array}{cc:cccc} \mu_{00} & 0 & 0 & 0 & \mu_{10} & \mu_{01} \\ 0 & \mu_{00} & -\mu_{10} & -\mu_{01} & 0 & 0 \\ \hdashline 0 & -\mu_{10} & \mu_{20} & \mu_{11} & 0 & 0 \\ 0 & -\mu_{01} & \mu_{11} & \mu_{02} & 0 & 0 \\ \mu_{10} & 0 & 0 & 0 & \mu_{20} & \mu_{11} \\ \mu_{01} & 0 & 0 & 0 & \mu_{11} & \mu_{02} \end{array}\right), \tag{2·8·8}$$

where μ_{pq} is the (p, q)th moment of $E(u, v)$ about $(\bar{u}, \bar{v})$. But since $(\bar{u}, \bar{v})$ is the centroid of E, the first-order moments μ_{10}, μ_{01} vanish. Hence ξ_3, ξ_4, ξ_5, ξ_6 are independent of ξ_1, ξ_2, and

$$p(\xi_1, \ldots, \xi_6) = p(\xi_1, \xi_2)\, p(\xi_3, \ldots, \xi_6), \tag{2·8·9}$$

where

$$p(\xi_3, \ldots, \xi_6) = \frac{1}{(2\pi)^2 \delta} \exp\{-[\mu_{02}(\xi_3^2 + \xi_5^2) - 2\mu_{11}(\xi_3\xi_4 + \xi_5\xi_6) + \mu_{20}(\xi_4^2 + \xi_6^2)]/2\delta\}, \tag{2·8·10}$$

and we have written

$$\delta = \begin{vmatrix} \mu_{20} & \mu_{11} \\ \mu_{11} & \mu_{02} \end{vmatrix}. \tag{2·8·11}$$

Now since

$$\xi_3 = \frac{\partial}{\partial x}(\rho \cos \phi) = \rho_x \cos \phi - \rho \phi_x \sin \phi, \tag{2·8·12}$$

etc.,

we have

$$\frac{\partial(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)}{\partial(\rho, \phi, \rho_x, \phi_x, \rho_y, \phi_y)} = -\rho^3, \tag{2·8·13}$$

and hence

$$p(\rho,\phi,\rho_x,\phi_x,\rho_y,\phi_y) = \frac{1}{(2\pi)^3\,m_{00}\,\delta}\,\rho^3\exp\{-\rho^2/2m_{00}\}$$
$$\times\exp\{-(\mu_{02}\rho_x^2-2\mu_{11}\rho_x\rho_y+\mu_{20}\rho_y^2)/2\delta\}\exp\{-\rho^2\{\mu_{02}\phi_x^2-2\mu_{11}\phi_x\phi_y+\mu_{20}\phi_y^2\}/2\delta\}. \quad (2{\cdot}8{\cdot}14)$$

From this distribution some immediate conclusions may be drawn. First, by integrating with respect to ϕ (from 0 to 2π) and ϕ_x, ϕ_y (from $-\infty$ to ∞) we obtain the joint distribution of ρ, ρ_x and ρ_y. Thus

$$p(\rho,\rho_x,\rho_y) = p(\rho)\,p(\rho_x,\rho_y), \quad (2{\cdot}8{\cdot}15)$$

where $p(\rho)$ is given by (2·8·6) and

$$p(\rho_x,\rho_y) = \frac{1}{2\pi\delta^{\frac{1}{2}}}\exp\{-(\mu_{02}\rho_x^2-2\mu_{11}\rho_x\rho_y+\mu_{20}\rho_y^2)/2\delta\}. \quad (2{\cdot}8{\cdot}16)$$

This shows that ρ_x, ρ_y are statistically independent of ρ, just as $\partial\zeta/\partial x$, $\partial\zeta/\partial y$ are independent of ζ. Further, the distributon (2·8·16) is formally identical with (2·1·12), if the moments μ_{pq} about the centroid are substituted for the moments m_{pq} about the origin. We deduce immediately that

(1) the steepest r.m.s. gradient of the envelope is in the principal direction of the envelope, and the gentlest r.m.s. gradient of the envelope is in the direction at right angles;

(2) the most probable direction of contours of the envelope is perpendicular to the principal direction of the envelope, and the least probable direction is parallel to the principal direction.

We see from (2·8·14) that the mean values of ϕ_x and ϕ_y are zero. It follows that *the phase angle ϕ of the envelope has zero secular increase in any horizontal direction.* Now the phase angle of ζ is the sum of the phase angles of the envelope and of the carrier wave. Hence *the phase angle of ζ increases at the same average rate as that of the carrier wave*, in any horizontal direction. This property is the result of our having chosen the centroid $(\overline{u},\overline{v})$ of the energy distribution as the wave-number of the carrier wave (§ 1·5).

By integrating (2·8·14) with respect to ρ_x, ρ_y and ϕ we obtain

$$p(\rho,\phi_x,\phi_y) = \frac{1}{2\pi m_{00}\,\delta^{\frac{1}{2}}}\,\rho^3\exp\{-\rho^2/2m_{00}\}\exp\{-\rho^2(\mu_{02}\phi_x^2-2\mu_{11}\phi_x\phi_y+\mu_{20}\phi_y^2)/2\delta\}. \quad (2{\cdot}8{\cdot}17)$$

This shows that ϕ_x and ϕ_y are *not* statistically independent of ρ. In fact the standard deviation of (ϕ_x,ϕ_y) (defined as the square root of the mean value of $(\phi_x^2+\phi_y^2)$) is

$$(\overline{\phi_x^2+\phi_y^2})^{\frac{1}{2}} = \frac{1}{\rho}(\mu_{20}+\mu_{02})^{\frac{1}{2}}, \quad (2{\cdot}8{\cdot}18)$$

which is inversely proportional to ρ. Roughly, this means that the higher waves are more regular than the lower waves (cf. § 2·10). The joint distribution of ϕ_x and ϕ_y alone is found from (2·8·17) to be

$$p(\phi_x,\phi_y) = \frac{1}{\pi}\,\frac{m_{00}/\delta^{\frac{1}{2}}}{[1+(\mu_{02}\phi_x^2-2\mu_{11}\phi_x\phi_y+\mu_{20}\phi_y^2)\,m_{00}/\delta]^2}. \quad (2{\cdot}8{\cdot}19)$$

It will be useful to consider also the statistical properties of the envelope of the curve in which the surface is intersected by a vertical plane in a direction θ. When $\theta = 0$, $x = x'$, the

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distribution of ρ, ϕ, ρ_x, ϕ_x may be found from (2·8·14) by integration with respect to ρ_y and ϕ_y. On replacing μ_{20} by $\mu_2(\theta)$ and x by x' we have in the general case

$$p(\rho,\phi,\rho_{x'},\phi_{x'}) = \frac{1}{(2\pi)^2 m_{00}\mu_2}\rho^2 \exp\{-\rho^2/2m_{00}\}\exp\{-(\rho_{x'}^2+\rho^2\phi_{x'}^2)/2\mu_2\}. \quad (2·8·20)$$

Alternatively, the distribution may be derived from first principles by the method used to obtain (2·8·14). The joint distribution of ρ and $\rho_{x'}$ is found by further integration with respect to ϕ and $\phi_{x'}$:

$$p(\rho,\rho_{x'}) = \frac{1}{(2\pi)^{\frac{1}{2}} m_{00}\mu_2^{\frac{1}{2}}}\rho \exp\{-\rho^2/2m_{00}\}\exp\{-\rho_{x'}^2/2\mu_2\}. \quad (2·8·21)$$

Similarly the joint distribution of ρ, ϕ and $\phi_{x'}$ is

$$p(\rho,\phi,\phi_{x'}) = \frac{1}{(2\pi)^{\frac{3}{2}} m_{00}\mu_2^{\frac{1}{2}}}\rho^2 \exp\{-\rho^2/2m_{00}\}\exp\{-\rho^2\phi_{x'}^2/2\mu_2\}, \quad (2·8·22)$$

and the distribution of ϕ and $\phi_{x'}$ is

$$p(\phi,\phi_{x'}) = \frac{(m_{00}/\mu_2)^{\frac{1}{2}}}{4\pi(1+\phi_{x'}^2 m_{00}/\mu_2)^{\frac{3}{2}}}. \quad (2·8·23)$$

From these distributions one can state immediately some general conclusions for the one-dimensional envelope analogous to those for the two-dimensional envelope of surface. Thus $\rho_{x'}$, but not $\phi_{x'}$, is independent of ρ; the mean secular increase of ϕ with x' is zero; the standard deviation of $\phi_{x'}$ is inversely proportional to ρ.

When the spectrum is fairly narrow, the envelope follows closely the crests of the waves. In any particular plane section the waves will appear in groups, and a rough measure of the average length of a group is given by $2/\mathbf{N}$, where $\mathbf{N}$ is the average number of times per unit distance that the envelope crosses an arbitrary level ρ. Now by the argument of § 2·2,

$$\mathbf{N}(\rho) = \int_{-\infty}^{\infty} p(\rho,\rho_{x'})\,|\,\rho_{x'}\,|\,\mathrm{d}\rho_{x'}. \quad (2·8·24)$$

On substituting from (2·8·23) and carrying out the integration we have

$$\mathbf{N}(\rho) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\mu_2^{\frac{1}{2}}}{m_{00}}\rho\exp\{-\rho^2/2m_{00}\}. \quad (2·8·25)$$

For definiteness we may take the largest possible value of $\mathbf{N}$, which occurs when $\rho = m_{00}^{\frac{1}{2}}$, giving

$$\mathbf{N} = \left(\frac{2}{e\pi}\right)^{\frac{1}{2}}\left(\frac{\mu_2}{m_{00}}\right)^{\frac{1}{2}}. \quad (2·8·26)$$

Now by § 1·5 $\mu_2(\theta)$ is greatest when θ defines the principal direction. It follows that *the average length of a group of waves is least in the principal direction and greatest in the direction at right angles.*

A rough measure of the number of waves in each group is given by $N_0/\mathbf{N}$, where N_0 is the number of zero-crossings of ζ in the direction θ (see § 2·2). When $\rho = m_{00}^{\frac{1}{2}}$ we have

$$\frac{N_0}{\mathbf{N}} = \left(\frac{e}{2\pi}\right)^{\frac{1}{2}}\left(\frac{m_2}{\mu_2}\right)^{\frac{1}{2}} = \left(\frac{e}{2\pi}\right)^{\frac{1}{2}}\left[\frac{m_{20}\cos^2\theta+2m_{11}\cos\theta\sin\theta+m_{02}\sin^2\theta}{\mu_{20}\cos^2\theta+2\mu_{11}\cos\theta\sin\theta+\mu_{02}\sin^2\theta}\right]^{\frac{1}{2}}. \quad (2·8·27)$$

In general this number will vary with the direction θ, but it may also be constant. The condition for constancy is

$$\mu_{20}:\mu_{11}:\mu_{02} = m_{20}:m_{11}:m_{02}. \quad (2·8·28)$$

When this condition is satisfied the number of waves in a group is independent of the direction.

If we write
$$\nu'(\theta) = \frac{m_{00}}{m_1(\theta)}\left(\frac{\mu_2(\theta)}{m_{00}}\right)^{\frac{1}{2}}, \tag{2·8·29}$$
so that (for a narrow spectrum)
$$\nu'(\theta) \doteqdot \left(\frac{\mu_2(\theta)}{m_2(\theta)}\right)^{\frac{1}{2}} = \left(\frac{2\pi}{e}\right)^{\frac{1}{2}}\frac{\mathbf{N}}{N_0}, \tag{2·8·30}$$
it is clear that ν' is inversely proportional to the number of waves in a group. In particular, when the section is taken in the direction $\theta = 0$ we have
$$\nu'(0) = \left(\frac{\mu_{20}}{\bar{u}^2 m_{00}}\right)^{\frac{1}{2}} = \nu, \tag{2·8·31}$$
where ν is the parameter defined in § 1·6. Now $\theta = 0$ was taken there to be the mean direction, and also the principal direction. It follows that ν *is inversely proportional to the number of waves in a group corresponding to a vertical section taken in the principal direction.*

2·9. *The heights of maxima*

Throughout this and the following section it will be assumed that the spectrum is narrow. We shall see that from the properties of the envelope one can then derive some interesting statistical properties that are otherwise difficult to obtain.

Consider first the distribution of the heights ξ of the crests. A crest may be defined as the locus of the maxima of all vertical sections of the surface parallel to the mean direction $\bar{\theta}$. Now when the spectrum is narrow, the waves will be long-crested and regular, and the crests will lie almost on the envelope. Further, the crests will be spaced at more or less equal intervals in the x, y plane. It follows that the distribution of the crest heights is practically the same as the distribution of the envelope function ρ. So from (2·8·6)
$$p(\xi) = \frac{\xi}{m_{00}}\exp\{-\xi^2/2m_{00}\}. \tag{2·9·1}$$
In other words, ξ has a Rayleigh distribution.

Consider, on the other hand, the distribution of the heights of the maxima. A maximum of the surface is simultaneously a maximum perpendicular and parallel to the mean direction. The distribution of maxima of the surface therefore approximates to the distribution of the maxima of the envelope of a section at right angles to the mean direction.

Now the distribution of the maxima of the envelope of a single random variable has been studied by Rice (1944, 1945). Making the simplifying assumption that $\mu_1 = \mu_3 = 0$ (in our case $\mu_1 = 0$ anyway), he obtains for the joint distribution of x' and the height R of a maximum
$$p(x', R) = \frac{1}{4\pi^{\frac{1}{2}}}\frac{\mu_2^{\frac{1}{2}}}{\mu_0}(a^2-1)^{\frac{3}{4}} z^{\frac{3}{2}} \mathrm{e}^{-a^2z^2} \sum_{n=0}^{\infty} \frac{A_n z^n}{(\frac{1}{2}n+\frac{3}{4})!}, \tag{2·9·2}$$
where
$$a^2 = \frac{\mu_0\mu_4}{\mu_2^2}, \quad z = \frac{R}{[2(a^2-1)\,\mu_0]^{\frac{1}{2}}}, \tag{2·9·3}$$
and
$$A_n = \sum_{m=0}^{n} \frac{(\frac{1}{2})(\frac{3}{2})\dots(m-\frac{1}{2})}{m!}(n-m+1)(\tfrac{3}{2}-\tfrac{1}{2}a^2)^m \tag{2·9·4}$$

(the term corresponding to $m = 0$ in (2·9·4) is $n+1$). To obtain the probability density of R alone we must normalize (2·9·2) by dividing by the number N of maxima per unit distance x'. N is found by integrating with respect to R from 0 to ∞. We have

$$N = \frac{1}{4(2\pi)^{\frac{1}{2}}} \left(\frac{\mu_2}{\mu_0}\right)^{\frac{1}{2}} \frac{(a^2-1)^2}{a^{\frac{5}{2}}} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}n+\frac{1}{4})!\,A_n}{(\frac{1}{2}n+\frac{3}{4})!\,a^n} \tag{2·9·5}$$

(Rice 1945, p. 83).

This may be checked immediately by comparison with our previous work. For, since the maxima occur only at the crests of the waves, which are more or less evenly spaced at distance $2\pi/\bar{u}$ apart, it follows that the mean density of maxima per unit area is $N\bar{u}/2\pi$ approximately. On replacing $(\mu_2/\mu_0)^{\frac{1}{2}}$ by $\gamma\bar{u}$, where γ^{-1} is the long-crestedness, we have

$$D_{\text{ma.}} = \frac{N\bar{u}}{2\pi} = \frac{1}{4(2\pi)^{\frac{3}{2}}} \frac{(a^2-1)^2}{a^{\frac{5}{2}}} \sum_{n=0}^{\infty} \frac{(\frac{1}{2}n+\frac{1}{4})!\,A_n}{(\frac{1}{2}n+\frac{3}{4})!\,a^n} \gamma\bar{u}^2. \tag{2·9·6}$$

This expression should agree with (2·3·57). Rice summed the series in (2·9·6) for $a^2 = 3$ (the normal distribution) and found it to be about 3·97. With this value (2·9·6) becomes

$$D_{\text{ma.}} = 0{\cdot}0638\gamma\bar{u}^2, \tag{2·9·7}$$

which is in agreement with the more accurate value $D_{\text{ma.}} = 0{\cdot}0639\gamma\bar{u}^2$ given by (2·3·57) and table 1.

The distribution of the heights of maxima may be stated in terms of the non-dimensional parameter $\eta = R/\mu_0^{\frac{1}{2}} = R/m_{00}^{\frac{1}{2}}$. On dividing $p(x', R)$ by $N\mu_0^{-\frac{1}{2}}$, we find

$$p(\eta) = \frac{1}{2^{\frac{1}{4}}(1-a^{-2})^{\frac{5}{4}}} \eta^{\frac{3}{2}} \exp\{-\eta^2/2(1-a^{-2})\} \sum_{n=0}^{\infty} \frac{A_n[\eta^2/2(a^2-1)]^{\frac{1}{2}n}}{(\frac{1}{2}n+\frac{3}{4})!} \Big/ \sum_{n=0}^{\infty} \frac{(\frac{1}{2}n+\frac{1}{4})!\,A_n}{(\frac{1}{2}n+\frac{3}{4})!\,a^n}. \tag{2·9·8}$$

This distribution has been computed for $a^2 = 2, 3, 5, 9$, and the curves are shown in figure 13.

When $a^2 \doteqdot 1$ the above series becomes unsuitable for computation, but we may obtain a solution by an independent method as follows. $a^2 = 1$ corresponds to two narrow bands of energy of slightly different frequency. These form beats, and the maxima of the envelope occur when the two wave bands are in phase. The height of the envelope R is then the sum of the amplitudes ρ_1, ρ_2 of the two wave trains at that point. But the amplitude of each wave train has a Rayleigh distribution:

$$p(\rho_1) = \frac{2\rho_1}{m_{00}} \exp\{-\rho_1^2/m_{00}\}, \qquad p(\rho_2) = \frac{2\rho_2}{m_{00}} \exp\{-\rho_2^2/m_{00}\} \tag{2·9·9}$$

(the mean energy for each wave train being $\frac{1}{2}m_{00}$). The distribution of the sum of these is

$$p(R) = \int_0^R p(\rho_1)\, p(\rho_2)\, \mathrm{d}\rho_1, \tag{2·9·10}$$

where $\rho_2 = R-\rho_1$. In terms of non-dimensional variables we have

$$p(\eta) = 2\int_0^\eta \xi\, \mathrm{e}^{-\xi^2}\, (\eta-\xi)\, \mathrm{e}^{-(\eta-\xi)^2}\, \mathrm{d}\xi. \tag{2·9·11}$$

On evaluating the integral we have

$$p(\eta) = \mathrm{e}^{-\frac{1}{2}\eta^2}\, [\eta\, \mathrm{e}^{-\frac{1}{2}\eta^2} + (\eta^2-1) \int_0^\eta \mathrm{e}^{-\frac{1}{2}t^2}\, \mathrm{d}t], \tag{2·9·12}$$

which is the distribution shown in figure 13 for $a^2 = 1$.

It can be seen independently that this is the appropriate distribution for two long-crested systems of waves intersecting at an angle. For the maxima of the combine system occur at the points of intersection of the crests of the two long-crested systems. The height of a maximum is the sum of the heights of the crests of the two systems. Consequently η is the sum of two variables each having a Rayleigh distribution.

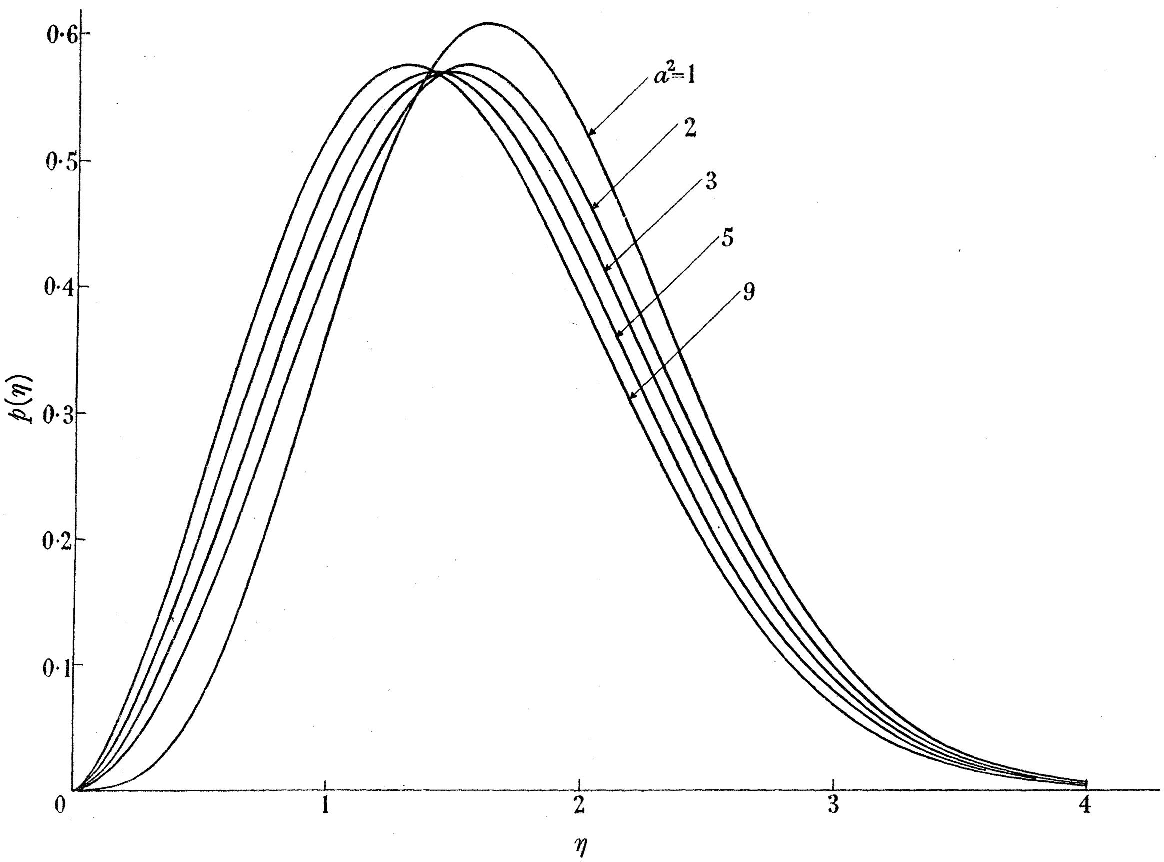

FIGURE 13. The probability distribution of the heights of maxima, for a narrow spectrum ($a^2 = 1, 2, 3, 5, 9$).

2·10. *The intervals between successive zeros*

Finally, let us consider the distribution of the intervals l between successive zeros of the surface, along a line drawn in an arbitrary direction θ. An approximate expression for the distribution of intervals for a one-dimensional function has been derived by Rice (1945, § 3·4) after a series of approximations assuming that the spectrum is narrow. It will now be shown how the same distribution can be derived very simply using the properties of the phase angle ϕ. Further, the distribution will be derived for a section of the surface in an arbitrary direction θ, and for waves of any particular amplitude ρ.

For simplicity we may take initially $\theta = \theta_p = 0$ and $x' = x$, and we may generalize to arbitrary values of θ at a suitable stage. The equation of the curve at an arbitrary time, say $t = 0$, may be written

$$\zeta = \mathscr{R}\rho\, e^{i\chi}, \qquad (2{\cdot}10{\cdot}1)$$

where

$$\chi = \overline{u}x + \phi, \quad \chi_x = \overline{u} + \phi_x, \qquad (2{\cdot}10{\cdot}2)$$

and $\bar{u} = m_1/m_0$. Like ϕ, χ is a multivalued function of x, having branches separated by multiples of 2π. Since

$$\frac{\partial(\chi, \chi_x)}{\partial(\phi, \phi_x)} = 1, \tag{2·10·3}$$

we have from (2·8·23)

$$p(\chi, \chi_x) = p(\phi, \phi_x) = \frac{(m_0/\mu_2)^{\frac{1}{2}}}{4\pi\{1+(\chi_x-\bar{u})^2\, m_0/\mu_2\}^{\frac{3}{2}}}. \tag{2·10·4}$$

Now ζ has a zero-crossing when and only when $\chi = n\pi$, where n is an integer. By the same reasoning as in § 2·2, the probability of χ taking the value $2r\pi$ in $(x, x+\mathrm{d}x)$ is

$$H(\chi)\,\mathrm{d}x = \int_{-\infty}^{\infty} p(\chi, \chi_x)\,|\,\chi_x\,|\,\mathrm{d}x\,\mathrm{d}\chi_x = \frac{1}{2\pi}\left(\frac{m_2}{m_0}\right)^{\frac{1}{2}}\mathrm{d}x. \tag{2·10·5}$$

The probability of χ taking the value $(2r+1)\pi$ is the same. Therefore the total probability of a zero in $(x, x+\mathrm{d}x)$ is twice this value, or

$$\frac{1}{\pi}\left(\frac{m_2}{m_0}\right)^{\frac{1}{2}}\mathrm{d}x, \tag{2·10·6}$$

in agreement with (2·3·5).

Let l denote the interval between successive zeros. The average value $\bar{l}$ of the distribution of l may be written down immediately; for it is simply the reciprocal of the average number of zeros per unit distance, i.e.

$$\bar{l} = \pi\left(\frac{m_0}{m_2}\right)^{\frac{1}{2}} = \frac{\pi}{\bar{u}}(1+\nu^2)^{\frac{1}{2}}, \tag{2·10·7}$$

where ν is defined by (2·8·31). When the spectrum is narrow (ν is small) we have

$$\bar{l} = \pi/\bar{u}, \tag{2·10·8}$$

approximately.

Let us now consider the whole distribution of l, on the assumption that ν is small. In the first place we may note that where χ crosses any level $n\pi$ it nearly always has an up-crossing. For the probability of a down-crossing ($\chi_x < 0$) in the interval $(x, x+\mathrm{d}x)$ is given by

$$\int_{-\infty}^{0} p(\chi, \chi_x)\,|\,\chi_x\,|\,\mathrm{d}x\,\mathrm{d}\chi_x = \frac{1}{4\pi}\left(\frac{m_2}{m_0}\right)^{\frac{1}{2}}[1-(1+\nu^2)^{-\frac{1}{2}}]\,\mathrm{d}x, \tag{2·10·9}$$

and the proportion of down-crossings is therefore

$$\tfrac{1}{2}[1-(1+\nu^2)^{-\frac{1}{2}}] \doteqdot \tfrac{1}{4}\nu^2, \tag{2·10·10}$$

which is negligible. If each crossing of $2r\pi$ or $(2r+1)\pi$ is an up-crossing, it follows that between any two successive zeros x_1 and x_2, χ must increase by π. Hence

$$\pi = \chi(x_2) - \chi(x_1) = [l\chi_x + \tfrac{1}{2}l^2\chi_{xx} + \dots], \tag{2·10·11}$$

where we have expanded in a Taylor series about $x = x_1$. It can be shown that χ_{xx} is of order ν^2, and so to our present order of approximation

$$l = \pi/\chi_x. \tag{2·10·12}$$

Now the distribution of χ_x, at points where χ takes a particular value, is given by

$$p(\chi_x)_\chi = \frac{p(\chi, \chi_x)\,|\,\chi_x\,|}{H(\chi)}, \tag{2·10·13}$$

where $H(\chi)$ is given by (2·10·5). That is to say

$$p(\chi_x)_\chi = \frac{1}{2}\left(\frac{m_0}{m_2}\right)^{\frac{1}{2}} \frac{|\chi_x|\,(m_0/\mu_2)^{\frac{1}{2}}}{[1+(\chi_x-\bar{u})^2\,m_0/\mu_2)]^{\frac{3}{2}}}. \qquad (2\cdot 10\cdot 14)$$

On substituting from (2·10·12) we have, when $l>0$,

$$p(l) = \frac{1}{2\pi}\left(\frac{m_0}{m_2}\right)^{\frac{1}{2}} \frac{\bar{u}^2/\nu}{[l^2/\bar{l}^2+(l/\bar{l}-1)^2/\nu^2]^{\frac{3}{2}}}, \qquad (2\cdot 10\cdot 15)$$

where $\bar{l}$ is given by the approximate relation (2·10·8). Clearly $p(l)$ is appreciable only when l differs from $\bar{l}$ by an amount of order ν. Writing

$$\xi = (l-\bar{l})/\bar{l} \qquad (2\cdot 10\cdot 16)$$

for the relative departure of l from its mean value, we have finally for the approximate distribution of ξ in the neighbourhood of the mean

$$p(\xi) = \frac{1}{2\nu(1+\xi^2/\nu^2)^{\frac{3}{2}}}. \qquad (2\cdot 10\cdot 17)$$

This is similar to the approximate distribution found by Rice (1945, p. 63) by a rather longer method.

In the general case when the line drawn on the surface is in an arbitrary direction θ, ν may be replaced by $\nu'(\theta)$. Thus we have in general

$$p(\xi) = \frac{1}{2\nu'(1+\xi^2/\nu'^2)^{\frac{3}{2}}}. \qquad (2\cdot 10\cdot 18)$$

The second moment of this distribution is divergent, but a convenient measure of its spread is the width of the interquartile range, given by

$$\frac{2}{\sqrt{3}}\nu'. \qquad (2\cdot 10\cdot 19)$$

So we may say that *the width of the distribution of ξ is inversely proportional to the average number of waves in a group*. The width of the distribution of l is given by the above expression multiplied by $\bar{l}$, that is to say

$$\frac{2\pi}{\sqrt{3}}\frac{m_0^{\frac{3}{2}}\mu_2^{\frac{1}{2}}}{m_1^2}. \qquad (2\cdot 10\cdot 20)$$

To find the distribution of intervals l for waves of a given amplitude (say with amplitude lying between ρ and $\rho+d\rho$), we may start from the distribution of (ϕ,ϕ_x) for values of ρ lying between these limits. This is given by

$$p_\rho(\phi,\phi_x) = \frac{p(\rho,\phi,\phi_x)}{p(\rho)} = \frac{1}{(2\pi)^{\frac{3}{2}}\mu_2^{\frac{1}{2}}}\rho\exp\{-\rho^2\phi_x^2/2\mu_2\} \qquad (2\cdot 10\cdot 21)$$

from (2·8·6) and (2·8·22). Hence

$$p_\rho(\chi,\chi_x) = \frac{1}{(2\pi)^{\frac{3}{2}}\mu_2^{\frac{1}{2}}}\rho\exp\{-\rho^2(\chi_x-\bar{u})^2/2\mu_2\}. \qquad (2\cdot 10\cdot 22)$$

On carrying out the same calculation with $p_\rho(\chi,\chi_x)$ in place of $p(\chi,\chi_x)$ we find, with ξ given by (2·10·16), that

$$p_\rho(\xi) = \frac{1}{(2\pi)^{\frac{1}{2}}\nu' m_{00}^{\frac{1}{2}}}\rho\exp\{-\rho^2\xi^2/2\nu'^2 m_{00}\}. \qquad (2\cdot 10\cdot 23)$$

This is a normal distribution for ξ, with mean value zero and standard deviation

$$\frac{\nu' m_{00}^{\frac{1}{2}}}{\rho}. \tag{2·10·24}$$

The mean and standard deviation of l are equal to $\bar{l}$ and $\nu' m_{00}^{\frac{1}{2}} \bar{l}/\rho$ respectively. Hence we may say that *the expectation of l is independent of the height of the waves* (to the present order of approximation) and also *the width of the distribution of l is inversely proportional to the wave height.* If we take this width as a measure of the irregularity of the waves as regards their intervals, we may also say that *the lower the waves, the less regular are their intervals.*

Part III. A method of determining the energy spectrum

In the two previous parts of this paper we have derived some statistical properties of a random surface in terms of its energy spectrum $E(u, v)$. In this part we solve the converse problem: given the statistical properties, to find the energy spectrum.

The best method of determining E depends to some extent upon which properties can be measured most conveniently. We assume that it is possible to obtain the height $\zeta(x')$ of the surface along a line in an arbitrary direction θ. (In the case of the sea surface one may imagine the observations to be made by an aircraft flying on a fixed course at high speed and constant altitude and recording by radar its height above the waves.) We also assume that $\partial\zeta(x')/\partial t$ can be measured (by a pair of radar sets, or otherwise).

In § 3·1 it is shown how from the statistical analysis of such measurements the moments $m_n(\theta)$, for each value of θ, can be deduced. In § 3·2 it is shown how to obtain the two-dimensional moments m_{pq} from the moments $m_n(\theta)$; and in § 3·3 how to obtain the energy spectrum from the moments m_{pq}.

3·1. *To obtain $m_n(\theta)$*

We saw in § 2·2 that the number of zeros of $\zeta(x')$ per unit horizontal distance x' is given by

$$N_0 = \frac{1}{\pi}\left(\frac{m_2(\theta)}{m_0(\theta)}\right)^{\frac{1}{2}}, \tag{3·1·1}$$

and in general the number of zeros of the rth derivative of ζ is given by

$$N_r = \frac{1}{\pi}\left(\frac{m_{2r+2}(\theta)}{m_{2r}(\theta)}\right)^{\frac{1}{2}}. \tag{3·1·2}$$

Now from the record of ζ, the numbers N_0, N_1, etc., may be determined by simple counting of zeros, maxima and minima, points of inflexion, and so on. $m_0(\theta)$ can be determined as the r.m.s. value of ζ along the curve. From (3·1·1) we have

$$m_2(\theta) = \pi^2 N_0^2 m_0(\theta) \tag{3·1·3}$$

and from (3·1·2)

$$m_{2r+2}(\theta) = \pi^2 N_r^2 m_{2r}(\theta). \tag{3·1·4}$$

So $m_2, m_4, \ldots, m_{2r+2}$ can be determined in succession, or else directly from

$$m_{2r+2}(\theta) = \pi^{2r+2} N_0^2 N_1^2 \ldots N_r^2 m_0(\theta). \tag{3·1·5}$$

To obtain the moments of odd order we have to use some property involving the motion of the surface. We take the distribution of the velocities of zeros of $\zeta(x')$, which was derived in § 2·5. It was shown that the mean velocity of the zeros of ζ is given by

$$\bar{c} = -\frac{m_1'(\theta)}{m_2(\theta)}, \tag{3·1·6}$$

and that the mean velocity of points where the rth derivative vanishes is

$$\bar{c}_r = -\frac{m'_{2r+1}(\theta)}{m_{2r+2}(\theta)}. \tag{3·1·7}$$

Now $m'_{2r+2}(\theta)$ is already known, from equation (3·1·5), so from

$$m'_{2r+1}(\theta) = -\bar{c}_r m_{2r+2}(\theta) \tag{3·1·8}$$

we may determine $m'_{2r+1}(\theta)$.

It is true that the odd moments $m'_{2r+1}(\theta)$ correspond not to the original function $E(u,v)$ but to $\sigma(u,v)\,E(u,v)$. However, we shall show in § 3·3 how this difficulty can be overcome.

3·2. *To obtain* m_{pq}

In § 1·4 we saw that $m_n(\theta)$ is related to the moments m_{pq} $(p+q=n)$ by the equation

$$m_n(\theta) = m_{n,0}\cos^n\theta + \binom{n}{1} m_{n-1,1}\cos^{n-1}\theta\sin\theta + \ldots + m_{0,n}\sin^n\theta. \tag{3·2·1}$$

The expression on the right-hand side is a trigonometric polynomial of degree n, with coefficients which are linear combinations of the moments. Therefore we may expect to solve for m_{pq} by taking the Fourier components of $m_n(\theta)$, that is, by considering the quantities

$$a_{n,l} = \frac{1}{\pi}\int_0^{2\pi} m_n(\theta)\,e^{il\theta}\,d\theta. \tag{3·2·2}$$

Going back to equation (1·4·11), we have

$$\begin{aligned} a_{n,l} &= \frac{1}{\pi}\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u,v)\,(u\cos\theta_1+v\sin\theta_1)^n\,du\,dv\,e^{il\theta_1}\,d\theta_1 \\ &= \frac{1}{\pi}\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u,v)\,\{w\cos(\theta-\theta_1)\}^n\,du\,dv\,e^{il\theta_1}\,d\theta_1, \end{aligned} \tag{3·2·3}$$

where $(w\cos\theta, w\sin\theta) = (u,v)$. On writing $\theta_1-\theta=\theta_2$ and reversing the order of integration we have

$$\begin{aligned} a_{n,l} &= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{2\pi} E(u,v)\,w^n\cos^n\theta_2\,e^{il(\theta+\theta_2)}\,d\theta_2\,du\,dv \\ &= \gamma_{n,l}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u,v)\,w^n\,e^{il\theta}\,du\,dv, \end{aligned} \tag{3·2·4}$$

where $\gamma_{n,l}$ is a numerical constant:

$$\gamma_{n,l} = \frac{1}{\pi}\int_0^{2\pi}\cos^n\theta_2\,e^{il\theta_2}\,d\theta_2 = \begin{cases} 2^{1-n}\binom{n}{r} & \text{when} \quad n-l=2r>0 \\ 0 & \text{otherwise.}\end{cases} \tag{3·2·5}$$

Now

$$\begin{aligned} m_{pq} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u,v)\,u^p v^q\,du\,dv \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u,v)\,w^n\cos^p\theta\sin^q\theta\,du\,dv \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u,v)\,w^n\frac{1}{2^n i^q}(e^{i\theta}+e^{-i\theta})^n(e^{i\theta}-e^{-i\theta})^q\,du\,dv \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(u,v)\,w^n\left[e^{in\theta}+\binom{p\quad q}{1}e^{i(n-2)\theta}+\binom{p\quad q}{2}e^{i(n-4)\theta}+\ldots+(-1)^q e^{-in\theta}\right]du\,dv, \end{aligned} \tag{3·2·6}$$

where $\begin{pmatrix} p & & q \\ & r & \end{pmatrix}$ is the coefficient of x^r in the expansion of $(1+x)^p\,(1-x)^q$. In full,

$$\begin{pmatrix} p & & q \\ & r & \end{pmatrix} = \binom{p}{r} - \binom{p}{r-1}\binom{q}{1} + \binom{p}{r-2}\binom{q}{2} - \ldots + (-1)^r \binom{q}{r}, \tag{3·2·7}$$

where $\binom{p}{0} = 1$ and $\binom{p}{r} = 0$ for $r > p$. From (3·2·4) and (3·2·6) we have

$$m_{pq} = \frac{1}{2\mathrm{i}^q}\left[a_{n,n} + \begin{pmatrix} p & & q \\ & 1 & \end{pmatrix}\Big/\binom{n}{1}\,a_{n,n-2} + \begin{pmatrix} p & & q \\ & 2 & \end{pmatrix}\Big/\binom{n}{2}\,a_{n,n-4} + \ldots + (-1)^q a_{n,n}\right]. \tag{3·2·8}$$

TABLE 2. FUNCTIONS $C_{pq}(\theta)$

$C_{00} = \frac{1}{2}$

$C_{10} = \cos\theta$
$C_{01} = \sin\theta$

$C_{20} = \cos 2\theta + \frac{1}{2}$
$C_{11} = \sin 2\theta$
$C_{02} = -\cos 2\theta + \frac{1}{2}$

$C_{30} = \cos 3\theta + \cos\theta$
$C_{21} = \sin 3\theta + \frac{1}{3}\sin\theta$
$C_{12} = -\cos 3\theta + \frac{1}{3}\cos\theta$
$C_{03} = -\sin 3\theta + \sin\theta$

$C_{40} = \cos 4\theta + \cos 2\theta + \frac{1}{2}$
$C_{31} = \sin 4\theta + \frac{1}{2}\sin 2\theta$
$C_{22} = -\cos 4\theta + \frac{1}{6}$
$C_{13} = -\sin 4\theta + \frac{1}{2}\sin 2\theta$
$C_{04} = \cos 4\theta - \cos 2\theta + \frac{1}{2}$

$C_{50} = \cos 5\theta + \cos 3\theta + \cos\theta$
$C_{41} = \sin 5\theta + \frac{3}{5}\sin 3\theta + \frac{1}{5}\sin\theta$
$C_{32} = -\cos 5\theta - \frac{1}{5}\cos 3\theta + \frac{1}{5}\cos\theta$
$C_{23} = -\sin 5\theta + \frac{1}{5}\sin 3\theta + \frac{1}{5}\sin\theta$
$C_{14} = \cos 5\theta - \frac{3}{5}\cos 3\theta + \frac{1}{5}\cos\theta$
$C_{05} = \sin 5\theta - \sin 3\theta + \sin\theta$

So on substitution from (3·2·2)

$$m_{pq} = \frac{1}{\pi}\int_0^{2\pi} m_n(\theta)\, C_{pq}(\theta)\,\mathrm{d}\theta, \tag{3·2·9}$$

where
$$C_{pq}(\theta) = \frac{1}{2\mathrm{i}^q}\left[\mathrm{e}^{\mathrm{i}n\theta} + \begin{pmatrix} p & & q \\ & 1 & \end{pmatrix}\Big/\binom{n}{1}\,\mathrm{e}^{\mathrm{i}(n-2)\theta} + \ldots + (-1)^n\,\mathrm{e}^{-\mathrm{i}n\theta}\right]. \tag{3·2·10}$$

The quantities $m_n(\theta)$ being known, this determines the moments m_{pq}. The first few functions $C_{pq}(\theta)$ are listed in table 2.

Incidentally, when the spectrum $E(u,v)$ has circular symmetry, $m_n(\theta)$ is independent of θ and so from (3·2·9)

$$m_{pq}(\theta) = \begin{cases} (-1)^{\frac{1}{2}q}\begin{pmatrix} p & & q \\ & \frac{1}{2}n & \end{pmatrix}\Big/\binom{n}{\frac{1}{2}n}\, m_n & \text{when } p, q \text{ are both even} \\ 0 & \text{otherwise.} \end{cases} \tag{3·2·11}$$

In particular
$$m_{20} = m_{02} = m_2, \qquad m_{40} = m_{04} = m_4, \qquad m_{22} = \tfrac{1}{3}m_4, \tag{3·2·12}$$

and so the condition (1·3·11) for a narrow ring spectrum reduces to

$$\tfrac{8}{3}m_4 m_0 - 4m_2^2 = 0 \tag{3·2·13}$$

or
$$\frac{m_4 m_0}{m_2^2} = \frac{3}{2}. \tag{3·2·14}$$

As a corollary, we see that $m_4 m_0/m_2^2$ is never less than $\frac{3}{2}$.

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3·3. *To obtain* $E(u,v)$

We have so far obtained the even moments m_{pq} of $E(u,v)$ and the odd moments m'_{pq} of $\sigma(u,v)\,E(u,v)$. Now consider the function

$$F(u,v) = \tfrac{1}{2}[E(u,v)+E(-u,-v)]. \tag{3·3·1}$$

This is clearly an even function of (u,v), since $F(-u,-v) = F(u,v)$. Therefore its odd moments vanish. But its even moments are the same as those of $E(u,v)$. Therefore both the odd and even moments of $F(u,v)$ are known. Similarly

$$G(u,v) = \tfrac{1}{2}[\sigma(u,v)\,E(u,v)-\sigma(-u,-v)\,E(-u,-v)] \tag{3·3·2}$$

is clearly an odd function of (u,v), since $G(-u,-v) = -G(u,v)$, and so its even moments vanish. But since $\sigma(-u,-v) = \sigma(u,v)$ (equation (1·1·5)) the odd moments of G are equal to those of $\sigma(u,v)\,E(u,v)$. Therefore both the odd and even moments of G are known. If F and G can both be determined from their moments we may then determine E, from the identity

$$E(u,v) = F(u,v)+G(u,v)/\sigma(u,v). \tag{3·3·3}$$

We have then simply to consider how to determine F and G from their moments.*

Formally, if the moments were known to all orders, the problem would be solved. For since the even moments m_{pq} are equivalent to the derivatives of the correlation function $\psi(x,y,0)$ (equation (1·2·10)) we have

$$\psi(x,y,0) = \sum_{p+q=2r}(-1)^r\frac{m_{pq}}{p!\,q!}x^p y^q. \tag{3·3·4}$$

But by (1·2·9), $\psi(x,y,0)$ is the cosine transform of $E(u,v)$ and so of $F(u,v)$. Hence

$$F(u,v) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(x,y,0)\cos(ux+vy)\,\mathrm{d}x\,\mathrm{d}y. \tag{3·3·5}$$

Similarly, if we define a function

$$\psi'(x,y,0) = \sum_{p+q=2r}(-1)^{r+1}\frac{m'_{pq}}{p!\,q!}x^p y^q, \tag{3·3·6}$$

we have

$$G(u,v) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi'(x,y,0)\sin(ux+vy)\,\mathrm{d}x\,\mathrm{d}y. \tag{3·3·7}$$

In practice, however, only a finite number of moments can be obtained, and if (3·3·4) is replaced by only a finite number of terms of the series, (3·3·5) does not converge. The problem then is to find a convergent sequence of approximations to F and G, each approximation depending on the moments of the function up to a finite order.

It was shown by Weierstrass (1885) that a function of a single variable may be approximated over a finite range by a polynomial, and that this may be done in a variety of different ways. A simple method is given by Courant & Hilbert (1953, § 4), which we generalize to two dimensions as follows. Consider the function†

$$g_n(u,v) = (1-u^2-v^2)^n. \tag{3·3·8}$$

* For a discussion of whether a function is uniquely determined by its moments, see Kendall (1952, chap. 4).

† This is different from the generalization suggested in Courant & Hilbert (1953, p. 68), and leads to a more homogeneous approximation.

As n tends to infinity, g_n tends to zero for all values of (u, v) inside the circle $u^2+v^2 = 1$ except the origin. Further, if S is any smaller circle of radius $\delta < 1$,

$$\iint_S g_n(u, v)\,\mathrm{d}u\,\mathrm{d}v = \int_0^{\delta}\int_0^{2\pi}(1-w^2)^n\,w\,\mathrm{d}w\,\mathrm{d}\theta = \frac{\pi}{n+1}\,[1-(1-\delta^2)^{n+1}], \tag{3·3·9}$$

which also tends to zero. However, the dominant part of the above integral is contributed by the neighbourhood of the origin, that is, if S' is any interior circle of fixed radius δ', however small, almost the entire contribution to the integral comes from S':

$$\lim_{n\to\infty}\frac{\iint_{S'} g_n(u, v)\,\mathrm{d}u\,\mathrm{d}v}{\iint_S g_n(u, v)\,\mathrm{d}u\,\mathrm{d}v} = \lim_{n\to\infty}\frac{1-(1-\delta'^2)^{n+1}}{1-(1-\delta^2)^{n+1}} = 1. \tag{3·3·10}$$

Now suppose that $f(u, v)$ is any continuous function of two variables that we wish to approximate in the region S, $(u^2+v^2)^{\frac{1}{2}} < \frac{1}{2}$. Then if (u, v) is any interior point of S, the function

$$f_n(u, v) = \frac{\iint_S f(u_1, v_1)\,[1-(u-u_1)^2-(v-v_1)^2]^n\,\mathrm{d}u_1\,\mathrm{d}v_1}{\iint_S [1-u_1^2-v_1^2]^n\,\mathrm{d}u_1\,\mathrm{d}v_1} \tag{3·3·11}$$

is a weighted mean of $f(u, v)$, with weighting function $g_n(u-u_1, v-v_1)$ centred on (u, v). And since the neighbourhood of (u, v) contributes almost all the weight when n is large we see that

$$\lim_{n\to\infty} f_n(u, v) = f(u, v). \tag{3·3·12}$$

The convenience of this approximation lies in the fact that $f_n(u, v)$ is a polynomial in (u, v) of degree $2n$, and with coefficients that are definite integrals taken over S. Further, if we assume that $f(u, v)$ is negligible or zero outside S the coefficients in $f_n(u, v)$ are simply combinations of the moments of f of order not greater than $2n$.

To apply the representation in the present case let us assume that $E(u, v)$ is negligible when $(u^2+v^2)^{\frac{1}{2}} > \frac{1}{2}w_0$, say. In other words, we assume a cut-off at high wave-numbers (some such assumption is in any case necessary in order to ensure the uniqueness of the solution.) Then we take as an approximation to $F(u, v)$

$$F_n(u, v) = \frac{n+1}{\pi w_0^2}\iint_{S_1} F(u_1, v_1)\,[1-(u-u_1)^2/w_0^2-(v-v)^2/w_0^2]^n\,\mathrm{d}u_1\,\mathrm{d}v_1, \tag{3·3·13}$$

where S_1 is the region $(u_1^2+v_1^2)^{\frac{1}{2}} \leqslant \frac{1}{2}w_0$. Similarly we take

$$G_n(u, v) = \frac{n+1}{\pi w_0^2}\iint_{S_1} G(u_1, v_1)\,[1-(u-u_1)^2/w_0^2-(v-v_1)^2/w_0^2]^n\,\mathrm{d}u_1\,\mathrm{d}v_1, \tag{3·3·14}$$

and finally

$$E_n(u, v) = F_n(u, v) + G_n(u, v)/\sigma(u, v). \tag{3·3·15}$$

On expanding the polynomial expressions in (3·3·13) and (3·3·14) and carrying out the integrations we find, say, for $n = 2$,

$$F_2(u, v) = \frac{3}{\pi}\,[m_{00}(w_0^2-u^2-v^2) - 2(m_{20}+m_{02})\,(w_0^2-u^2-v^2)\,w_0^2 + (m_{40}+m_{04}+2m_{22})\,w_0^4] \tag{3·3·16}$$

and

$$G_2(u,v) = \frac{3}{\pi}\left[4(m'_{10}u + m'_{01}v)\,(w_0^2 - u^2 - v^2)\,w_0 - 4\{(m'_{30} + m'_{12})\,u + (m'_{21} + m'_{03})\,v\}\,w_0^3\right], \quad (3{\cdot}3{\cdot}17)$$

so that F_2 and G_2 are expressible as polynomials in (u, v) having as coefficients the moments of E up to degree 4. Approximations of higher order may be written down at will.

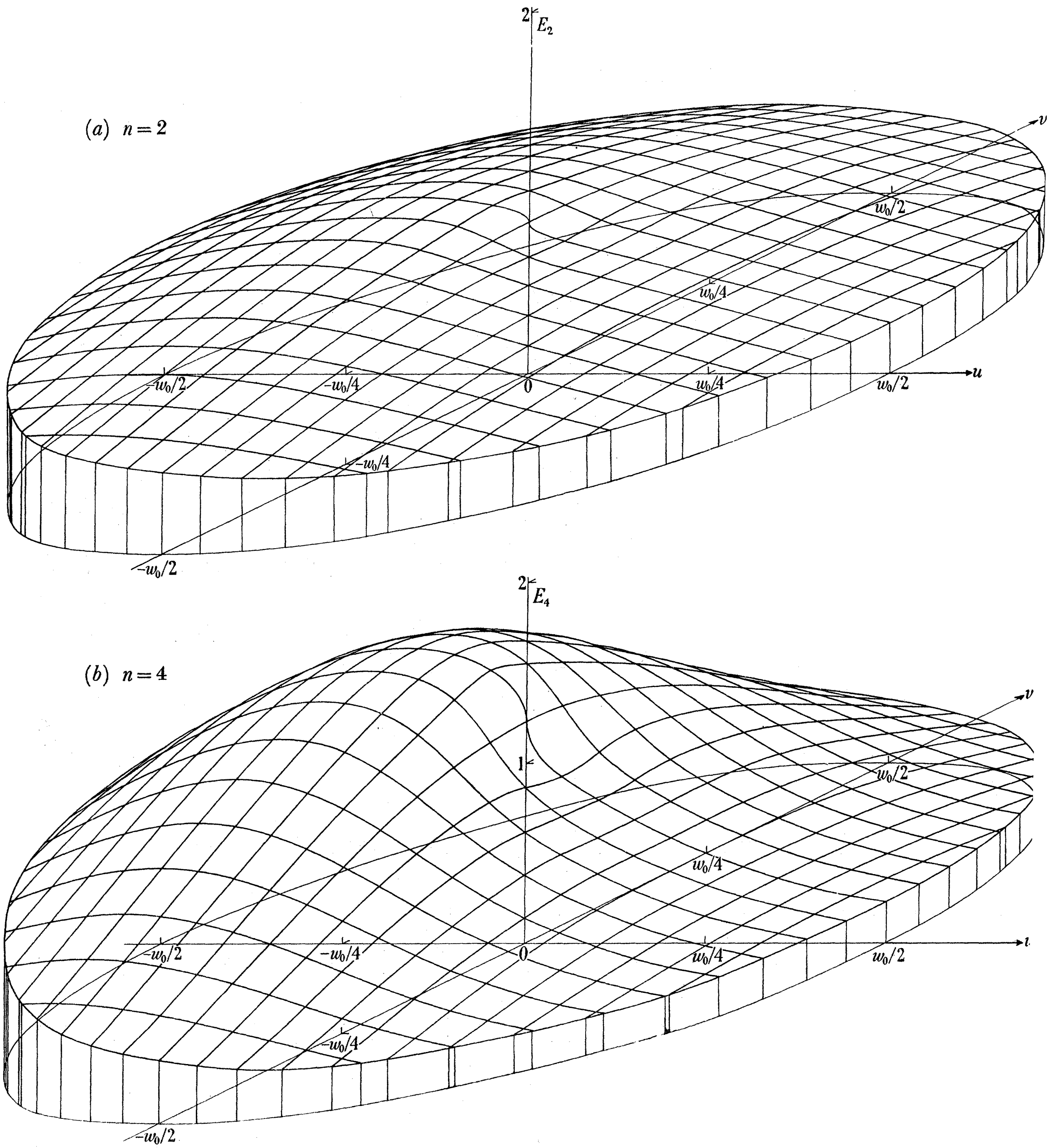

FIGURE 14

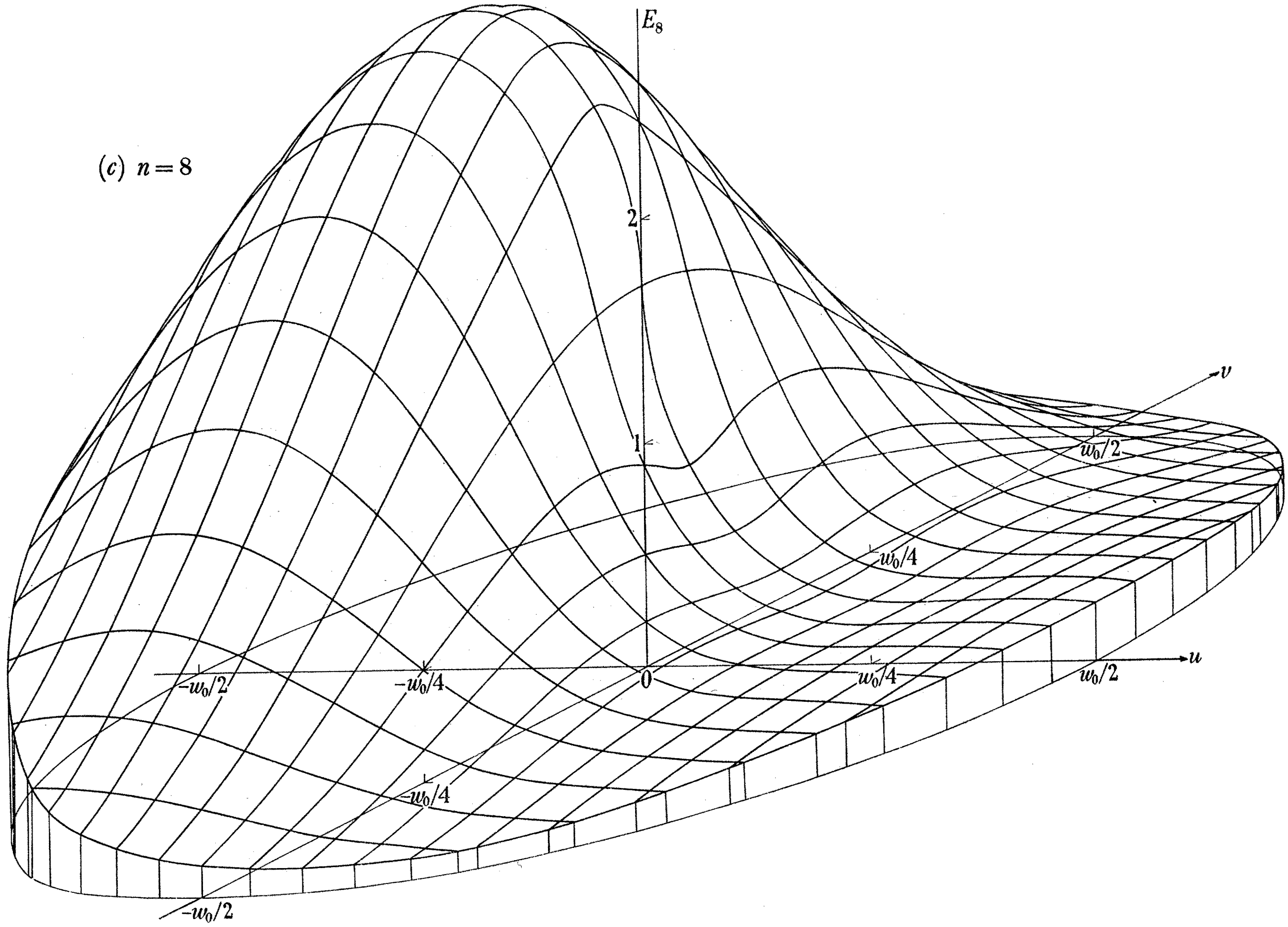

FIGURE 14. Successive approximations $E_n(u,v) = W_n(u, v; w_0/4, 0)$.

We have seen that F_n and G_n are essentially weighted averages of F and G by a weighting function proportional to $g_n[(u-u_1)/w_0, (v-v_1)/w_0]$. The weighting function corresponding to E_n is somewhat different owing to the presence of the factor $\sigma(u,v)$ in (3·3·15). In fact we have from (3·3·1) and (3·3·2)

$$E_n(u,v) = \frac{n+1}{2\pi w_0^2}\iint_{S_1} E(u_1,v_1)\left[1+\frac{\sigma(u_1,v_1)}{\sigma(u,v)}\right]\left[1-\left(\frac{u-u_1}{w_0}\right)^2-\left(\frac{v-v_1}{w_0}\right)^2\right]^n \mathrm{d}u_1\,\mathrm{d}v_1$$
$$+\frac{n+1}{2\pi w_0^2}\iint_{S_1} E(-u_1,-v_1)\left[1-\frac{\sigma(u_1,v_1)}{\sigma(u,v)}\right]\left[1-\left(\frac{u+u_1}{w_0}\right)^2-\left(\frac{v+v_1}{w_0}\right)^2\right]^n \mathrm{d}u_1\,\mathrm{d}v_1. \quad (3·3·18)$$

On changing the sign of (u_1, v_1) in the second integral we have

$$E(u,v) = \frac{1}{w_0^2}\iint_{S_1} E(u_1,v_1)\, W_n(u,v;u_1,v_1)\,\mathrm{d}u_1\,\mathrm{d}v_1, \quad (3·3·19)$$

where
$$W_n(u,v;u_1,v_1) = \frac{n+1}{2\pi}\left[1+\frac{\sigma(u_1,v_1)}{\sigma(u,v)}\right]\left[1-\left(\frac{u-u_1}{w_0}\right)^2-\left(\frac{v-v_1}{w_0}\right)^2\right]^n$$
$$+\frac{n+1}{2\pi}\left[1-\frac{\sigma(u_1,v_1)}{\sigma(u,v)}\right]\left[1-\left(\frac{u+u_1}{w_0}\right)^2-\left(\frac{v+v_1}{w_0}\right)^2\right]^n. \quad (3·3·20)$$

W_n is not a function of $(u-u_1)$ and $(v-v_1)$ alone. However, the second half of (3·3·20) only gives a contribution when $(u_1, v_1) \doteqdot (u, v)$, and then this is small owing to the presence of the factor $[1-\sigma(u_1, v_1)/\sigma(u, v)]$.

To obtain an idea of the accuracy of successive approximations we may consider the case of a narrow spectrum, when $E(u, v)$ is appreciably large only in the neighbourhood of a single point, say $(-\frac{1}{4}w_0, 0)$. Then

$$E_n(u, v) = W_n(u, v; -\tfrac{1}{4}w_0, 0). \tag{3·3·21}$$

W_n has been computed for $n = 2, 4, 8$ assuming that, as for gravity waves on deep water,

$$\sigma(u, v) \propto (u^2+v^2)^{\frac{1}{4}}. \tag{3·3·22}$$

The results are shown in figure 14 *a*, *b* and *c*. It will be seen how the functions become progressively more peaked as the degree of the approximation is raised. When $n = 8$ the area in which W_n exceeds half its maximum value has a radius of about $0·3w_0$. For large values of n we have, in the neighbourhood of (u_1, v_1),

$$W_n \doteqdot \frac{n}{\pi}\exp\{-n[(u-u_1)^2+(v-v_1)^2]/w_0^2\}, \tag{3·3·23}$$

and so the 'radius' of W_n is proportional to $n^{-\frac{1}{2}}$. It will be seen then that E_n converges to E rather slowly. In order to distinguish parts of the spectrum separated by a distance δ, it is necessary to take n to be of order $(w_0/\delta)^2$.

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